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Majority or Majorities?

by

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1. Introduction

It is common to read sentences like "x is better than y on a majority of criteria" or "a majority of voters prefer x to y". What do such sentences precisely mean? Although the concept of majority, in everyday life, seems unproblematic and well understood by most people, some difficulties arise when we want to use it formally in MCDA, like in ELECTRE (Roy and Bouyssou, 1993), Melchior (Leclercq, 1984), PROCFTN (Belacel and Boulassel, 2004), TACTIC (Vansnick, 1986), VIKOR (Opricovic and Tzeng, 2004) or VIP-G (Dias and Climaco, 2005). The reason of these difficulties is mainly that it is not clear what the statement "x is better than y" means when, for some criteria, indifference is allowed. Suppose indeed that we have the following situation: x strictly better than y on three criteria, y strictly better than x on two criteria and x and y indifferent on two criteria. If we look only at strict preferences, then x is better than y on a majority of criteria (3 out of 5) but, if we look at all criteria, then x is better than y on a minority of criteria (3 out of 7). The problem is even more complex when qualified majorities, i.e. majorities with a threshold, are used. So, depending on the threshold that we use and on the way we take indifferences into account, there is not one majority but many : simple majority, absolute majority, weak majority, ... The aim of this paper is to present and compare some of them.

In section 2, we will introduce some notation and present some majorities that will be analyzed subsequently. We will limit our analysis to non-weighted majorities because our current understanding of weighted majorities is still limited and does not allow us to analyze all weighted majorities in a unified framework. Yet, we hope that a sound analysis of non-weighted majorities can help to enhance our understanding of the corresponding weighted majorities. We will also limit our analysis to neutral majorities. Non-neutral majorities are very common in committees and parliaments, where a proposition is often opposed to the status quo; If the proposition has the support of a majority (e.g. at least 60% of the deputees attending vote for the proposition), then the proposition passes. Otherwise, it is rejected (i.e. the status quo passes). With such a procedure, the two alternatives (the proposition and the status quo) are not treated equally: the status quo can pass with less than 60% of support. We say that such a procedure is not neutral. In multicriteria decision aiding, very often, we want to treat all alternatives equally. That is why we restrict our attention to neutral majorities. The reader interested in non-neutral majorities will have a look at (Fishburn,1973).

Section 3 will be devoted to the analysis of the majorities presented in section 2. We will show what they have in common but also what makes them different, what are their salient characteristics. With this information at hand, we hope that analysts or people designing new decision aiding techniques will be able to choose a majority that is adequate for their problem (if they want to use a majority at all). Section 4 will conclude.

The present paper is based on a more technical one (Marchant, 2005). The interested reader will also have a look at the abundant literature on voting theory. A good starter for this might be (Bouyssou et al., 2000, ch. 2) and (Bouyssou et al., 2006, ch. 5). Not to miss on simple majority: May, 1952.

2. An overview of the main majority rules

Suppose we have a decision problem for which we think n criteria are relevant. We represent these criteria by n natural numbers: 1, 2, …, n. And the set of criteria is denoted by N. In this paper, we consider problems in which, for each criterion, we have a preference relation, denoted by $\geq_i$, on the set of alternatives (finite and denoted by X). These preference relations can be directly expressed by the decision-maker or can be derived from a performance table using some preferential information or … but this is not the focus of this paper and we will assume that the preference relations are given. The statement "$x \geq_i y$" means "x is at least as good as y on criterion i". If $x \geq_i y$ and NOT $y \geq_i x$, then x is strictly better than y and we write $x >_i y$. If $x \geq_i y$ and $y \geq_i x$, then x is indiferent to y and we write $x =_i y$. The n-uple $(\geq_i)_{i\in N}$ is called a profile and denoted by $\geq_N$. It is used to represent the preferences of a decision-maker according to n criteria.

Given a profile $\geq_N$, one may try to construct a global preference relation. This we do by means of an aggregation procedure (denoted by $\geq$), i.e. a function mapping each profile $\geq_N$ on a global preference relation denoted by $\geq(\geq_N)$. When there is a strict preference (resp. an indifference), we will use the symbol $>$ (resp. $\sim$) We can think of many different aggregation procedures. For instance, the plurality rule. It ranks the alternatives according to the number of criteria for which they are ranked first.

Example 1. Suppose $X=\{w, x, y, z\}$ and $N=\{1, 2, 3\}$. Suppose also that the preferences of the decision-maker are the following weak order[1]: $x >_1 y =_1 w >_1 z$, $x >_2 z >_2 w >_2 y$, $y >_2 w >_3 z =_3 x$. Alternative x (resp. y) is ranked first on criteria 1 and 2 (resp. 3). According to the plurality rule, the global preference relation is this weak order: $x >(\geq_N) y >(\geq_N) w \sim(\geq_N) z$. When only one profile is under consideration and no confusion is possible, we just write $x > y > w \sim z$.

We can also use the anti-plurality: It ranks the alternatives in decreasing order of the number of criteria for which they are ranked last. Using the profile of Example 1, we obtain the following weak order: $w > x \sim y > z$. Note that it is different from the plurality ranking. There are many other rules: the Borda rule, dictatorial rules, Kemeny rule, … and, of course, the majority rules that we now present. Let $P(x,y,\geq_N)$ be the number of criteria in N for which $x >_i y$ and $I(x,y,=_N)$ be the number of criteria in N for which $x =_i y$. Of course, $P(x,y,\geq_N) + I(x,y,=_N) + P(y,x,\geq_N) = n$. We are now ready to present some important majority rules.

Weak Majority. With this rule, x is globally at least as good as y iff x is at least as good as y on half the number of criteria. Since n can be odd, we need to be careful in the formal definition of the rule: $x \geq(\geq_N) y$ iff $P(x,y,\geq_N) + I(x,y,=_N) \geq \lceil n/2 \rceil$, where $\lceil t \rceil$ is the smallest integer not smaller than t (upwards rounding). If we apply this rule to the profile of Example 1, we obtain yet another weak order: $x > y \sim w > z$.

Qualified Weak Majority. This rule is similar to the previous one but uses a threshold possibly smaller than $\lceil n/2 \rceil$: $x \geq(\geq_N) y$ iff $P(x,y,\geq_N) + I(x,y,=_N) \geq \delta$, whith δ integer and $0 < \delta \leq \lceil n/2 \rceil$. It is of course possible to choose δ non-integer but several values of δ then lead to the same aggregation procedure (e.g. 2.1, 2.3 and 2.9). If we apply this rule to the profile of Example 1 with $\delta = 1$, we obtain yet another weak order: $x \sim y \sim w \sim z$; all alternatives are indifferent. Actually, qualified weak majority is not a single aggregation procedure, but a family of procedures, depending on the parameter δ. This family includes weak majority.

ELECTRE I Majority. This rule is similar to the previous one but uses a threshold possibly larger than $\lceil n/2 \rceil$: $x \geq(\geq_N) y$ iff $P(x,y,\geq_N) + I(x,y,=_N) \geq \delta$, whith δ integer and $\lceil n/2 \rceil \leq \delta \leq n$. If we apply this rule to the profile of Example 1 with $\delta = 3$, we find that all alternatives are incomparable. This is the aggregation procedure used in ELECTRE I to construct the concordance relation, eventually with weights. ELECTRE I is also a family of procedures, including weak majority.

Remark that qualified weak majority and ELECTRE I majority can be seen as special cases of a larger family where the threshold can vary in $]0,n]$. We suggest to call this family generalized qualified weak majority.

Simple Majority. This rule is quite different from the previous ones. Instead of comparing the weak support of x (i.e. $P(x,y,\geq_N) + I(x,y,=_N)$) with a threshold, it compares the weak support of x with the weak support of y (i.e. $P(y,x,\geq_N) + I(x,y,=_N)$). Formally, $x \geq(\geq_N) y$ iff $P(x,y,\geq_N) + I(x,y,=_N) \geq P(x,y,\geq_N) + I(x,y,=_N)$. Note that $I(x,y,=_N)$ cancels out on both sides of the equation but we keep it to make clear the link with a rule that we will introduce later. If we apply this rule to the profile of Example 1, we find the same weak order as with the weak majority.

a-Qualified Simple Majority. The rules in this family are similar to the previous one. They also compare the weak supports of x and y, but use an additive threshold. Formally, $x \geq(\geq_N) y$ iff $P(x,y,\geq_N) + I(x,y,=_N) \geq P(x,y,\geq_N) + I(x,y,=_N) + \delta$, with δ integer and $-n < \delta \leq n$. If we apply this rule to the profile of Example 1 with $\delta = 1$, we find a relation that is not complete: $x > y > z$, $x > w > z$ and $x > z$ but w and y are incomparable. Simple majority is of course a special case of a-qualified simple majority.

m-Qualified Simple Majority. This family, contrary to the previous one, uses a multiplicative

[1] A weak order is a complete and transitive relation. It is a complete ranking, possibly with ties.

threshold. Formally, $x \geq(\geq_N)\ y$ iff $P(x,y,\geq_N) + I(x,y,=_N) \geq \delta\ [\ P(x,y,\geq_N) + I(x,y,=_N)\ ]$, with $0 < \delta \leq n$ and $k\delta$ in N for some k in N. If we apply this rule to the profile of Example 1 with $\delta = 1.5$, we find again a relation that is not complete: all alternatives are incomparable except that $x > z$. Simple majority is of course a special case of m-qualified simple majority.

Tactic Majority. This family, like the previous one, uses a multiplicative threshold but it is based on the strict support (i.e. $P(x,y,\geq_N)$) and not on the weak one. Formally, $x >(\geq_N)\ y$ iff $P(x,y,\geq_N) > \delta\ P(x,y,\geq_N)$, with $1 \leq \delta < n$ and $k\delta$ in N for some k in N; otherwise, x and y are incomparable. If we apply this rule to the profile of Example 1 with $\delta = 2$, we find the same relation as with the *a-Qualified Simple Majority* with $\delta = 1$.

Note that the many families we introduced are distinct only if the single-criterion preferences contain some indifferences. Otherwise, the following three families are equivalent: generalized qualified weak majority, a-qualified simple majority and m-qualified simple majority. TACTIC majority is almost equivalent to these families; the only difference being that any indifference in the global preference relation becomes an incomparability with TACTIC.

3. Analysis of the main majority rules.

The many different majorities that we have presented in section 2 are distinct aggregation procedures: they sometimes lead to different global preference relations. Before using one or the other, it is therefore important to know what makes them different, what the distinctive properties of each one is. Only then is it possible to choose with the full knowledge of the facts. Nevertheless, these procedures also have a lot in common. So, before presenting the distinctive properties, we will show what these procedures share.

With all majority rules of Section 2, the rule used to determine the preference (>,< or ~) between two alternatives is the same for (x,y), (x,z), (y,z), (w,z), etc. All aternatives are treated in the same way. This is called *neutrality*.

With all majority rules of Section 2, all criteria play exactly the same role. Indeed, the global preference relation depends only on the numbers of criteria for which $x >_i y$, $x <_i y$ or $x =_i y$, but not on the criteria themselves. This property is called *anonymity*.

It is also clear that, with all majority rules of section 2, the global preference relation between x and y depends only on $P(x,y,\geq_N)$ and $I(x,y,=_N)$. We do not need to know anything about z or w for determining the global preference between x and y. This is called *independence of irrelevant alternatives*. Note that it is not satisfied by the plurality rule.

The following two conditions are monotonicity conditions and are satisfied by all majority rules of section 2. Suppose that, using the aggregation procedure $\geq$ with the profile $\geq_N$, we find $x \geq(\geq_N)\ y$. Suppose now that alternative x (say an investment plan) is improved relatively to y, in some way, on criterion i. On the other criteria nothing changes. Since x was globally as good as x before the improvement, it should still be as good as x after the improvement (and eventually better than x). An aggregation procedure respecting this principle is said to satisfy *weak non-negative responsiveness*. Since we did not define what an improvement is, weak non-negative responsiveness is not yet well-defined. We consider two kinds of improvements: going from $y >_i x$ to $x >_i y$ and from $x =_i y$ to $x >_i y$. The first one corresponds to *weak non-negative responsiveness 1*; the second one to *weak non-negative responsiveness 2*.

The last property that all majority rules of section 2 share is *unanimity*: when x is strictly better than y on all criteria, then x is globally strictly better than y. Note that it is not satisfied by the plurality rule.

Since it is hard to conceive a majority rule that would not satisfy independence of irrelevant alternatives, non-negative responsiveness 1 and 2 or unanimity, we propose the following definition for a majority rule: an aggregation procedure is a majority rule if and only if it satisfies independence of irrelevant alternatives, non-negative responsiveness 1 and 2 and unanimity. If, in addition, it satisfies neutrality and anonymity, we then say that it is a symmetric majority rule.

So, if we find the above-mentioned properties compelling, in a particular decision problem, then, we should probably use a symmetric majority. But, which one? We try to answer this question by providing, for each symmetric majority rule (or family of symmetric majority rules) of section 2, one or two properties that only that rule satisfies.

Generalized qualified weak majority.

A distinctinctive property of generalize qualified weak majority is *limited influence of indifference*. It is the same property as weak non-negative responsiveness 2 except that it is stated for a deterioration instead of an improvement. More precisely, suppose that, using the aggregation procedure $\geq$ with the profile $\geq_N$, we find $x \geq(\geq_N) y$. Suppose now that alternative x (say an investment plan) is deteriorated relatively to y, going from $x >_i y$ to $x =_i y$, on criterion i. On the other criteria nothing changes. Since x was globally weakly preferred to y before the deterioration and since the weak support of x against y does not change, it should still be weakly better than y after the deterioration[2]. An aggregation procedure respecting this principle is said to satisfy limited influence of indifference. All generalized qualified weak majority rules satisfy limited influence of indifference and no other symmetric majority rule satisfies it. So, if we find the common properties (neutrality, anonymity, independence of irrelevant alternatives, weak non-negative responsiveness 1 and 2, unanimity) compelling and if we think limited influence of indifference is also an important property, then we must use one of the generalized qualified weak majority rules.

If, in addition, we think that incomparability should be allowed, then we must use an ELECTRE I majority rule. If, on the contrary, we want that the global preference relation be complete (i.e. without incomparabilities), then we must use a weak qualified majority rule. Finally, if we want a complete relation but, with as few indifferences as possible, then we must use the weak majority rule.

a-Qualified simple majority.

A distinctive property of the a-qualified simple majority is *pairwise cancellation*. Suppose that, using the aggregation procedure $\geq$ with the profile $\geq_N$, we find $x \geq(\geq_N) y$. Suppose now that alternative x is deteriorated relatively to y, going from $x >_i y$ to $x =_i y$, on criterion i. Suppose also that alternative y is deteriorated relatively to x, going from $y >_j x$ to $x =_j y$, on criterion j. On the other criteria nothing changes. One can argue that the deterioration of x on criterion i exactly compensates the deterioration of y on criterion j. The two deteriorations cancel out each other. Hence, since x was globally weakly preferred to y before the changes, it should stilll be so after the changes. A similar argument can be used when x is improved relatively to y, going from $x =_i y$ to $x >_i y$, on criterion i and y is improved relatively to x, going from $y =_j x$ to $y >_j x$, on criterion j. An aggregation procedure respecting this principle is said to satisfy pairwise cancellation. All a-qualified majority rules satisfy pairwise cancellation and no other symmetric majority rule satisfies it.

If, in addition, we think that incomparability should be allowed, then we must use an a-qualified simple majority rule with a positive threshold. If, on the contrary, we want that the global preference relation be complete (i.e. without incomparabilities), then we must use a non-positive threshold. Finally, if we want a complete relation but, with as few indifferences as possible, then we must use the simple majority rule.

Note that, depending on the sign of the threshold, an a-qualified simple majority will always yield global preference relations possibly containing indifferences ($\delta \leq 0$) or incomparabilities ($\delta > 0$) but never both (contrary to ELECTRE I majority).

Tactic majority.

A distinctive property of the TACTIC majority rule is *P-invariance*. Suppose we have two profiles $\geq_N$ and $\geq'_N$ such that

One can argue that, since the ratio of the strict supports is the same in both profiles, the outcome should also be the same; i.e. $x \geq(\geq_N) y$ iff $x \geq(\geq'_N) y$. An aggregation procedure respecting this principle is said to satisfy P-invariance. Another distinctive property is *asymmetry*: there are no indifferences. All TACTIC majority rules satisfy P-invariance and asymmetry and no other symmetric majority rule satisfies both properties.

m-Qualified simple majority.

A distinctive property of the a-qualified simple majority is *PI-invariance*. It is similar to P-invariance except that it is based on weak supports instead of strict ones. Formally, suppose

$$\frac{P(x,y,\geq_N)+I(x,y,\geq_N)}{P(y,x,\geq_N)+I(y,x,\geq_N)} = \frac{P(x,y,\geq'_N)+I(x,y,\geq'_N)}{P(y,x,\geq'_N)+I(y,x,\geq'_N)}$$

One can argue that, since the ratio of the weak supports is the same in both profiles, the outcome should also be the same; i.e. $x \geq(\geq_N) y$ iff $x \geq(\geq'_N) y$. An aggregation procedure respecting this principle is said to satisfy PI-invariance. All m-qualified simple majority rules satisfy it and no other symmetric majority rule satisfies it.

4. Discussion.

We have seen that each family of symmetric majority rules can be distinguished from the others by using a single property: limited influence of indifference, pairwise cancellation, P-invariance or

[2] Note that, if $x \succ(\geq_N) y$ before the deterioration, then limited influence of indifference does not exclude that $x \sim(\geq_N) y$. So, indifference can have an influence on the result of the aggregation but it is limited.

PI-invariance. Each of these properties has a similar structure: it starts with a profile $\geq_N$ then considers a second profile $\geq'_N$, similar to the first one but where the relative position of *x* and *y* has been modified in a specific way. The property finally imposes that the global preference relation between *x* and *y* should be the same because the two profiles are very similar. Suppose an analyst wants to use a symmetric majority rule but is wondering which one to use. Since each family corresponds to one specific property, the analyst may well concentrate on these distinctive properties and forget the rest. If he thinks that one of these properties makes more sense than another one, in his decision context, then he automatically knows which aggregation procedure to use.

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