Stability of efficient solutions against weight changes in multi-objective linear programming models

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1 Introduction

Decision making problems, essentially those stemming from complex and ill-structured situations, inherently include the existence of multiple, conflicting and incommensurate aspects of evaluation of the different courses of action. In this work, some experimental studies are presented concerning the stability of efficient solutions regarding weighting vector changes in MOLP problems. The computational tools have been implemented as an extension of the visual interactive tolerance approach to sensitivity analysis proposed in [1, 2]. These are based on the analysis of the indifference regions on the parametric diagram associated with basic efficient solutions, which are computed by solving a scalar optimisation problem consisting of a non-negative weighted sum of the objective functions.

2 Tolerance approach to sensitivity analysis

One of the main drawbacks of traditional sensitivity analysis in LP problems is that the ranges obtained for the changes of the different coefficients, maintaining the same optimal basis, are only easily determined when the coefficients are not allowed to change in a simultaneous manner.

The tolerance approach to sensitivity analysis developed by Wendell [4, 5], for single objective LP problems enables to consider the impact of simultaneous and independent changes of more than one coefficient. Extensions for multi-objective problems have been proposed by Hansen *et al.* [3] and in the visual interactive tolerance approach to sensitivity analysis [1, 2].

In particular, the techniques proposed in [1, 2, 3] allow to study the behaviour of basic efficient solutions subject to simultaneous and independent changes of more than one component of the weighting vector. Perturbations on the objective function weights are considered and the largest percentage, called the maximum tolerance percentage, by which all weights can deviate simultaneously and independently from their estimated values while retaining the same efficient basic solution is computed.

3 Some illustrative results

Let us consider the following MOLP problem (also used in [2, 3])

$$\begin{aligned}
\max z_1 &= 10\,x_2 + 80\,x_4 \\
\max z_2 &= 10\,x_2 + 10\,x_3 + 20\,x_4 \\
\max z_3 &= 10\,x_1 + 10\,x_2 + 10\,x_3 + 10\,x_4 \\
\text{s.a.} & \\
& 4\,x_1 + 9\,x_2 + 7\,x_3 + 10\,x_4 \leq 6000 \qquad (\mathbf{c_1}) \\
& x_1 + x_2 + 3\,x_3 + 40\,x_4 \leq 4000 \qquad (\mathbf{c_2}) \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}$$

Table 1: Information about the basic efficient solutions.

Solution	z_1	z_2	Z_3	Area (%)	$\mathbf{x}_B$
1	12571.40	7428.57	6571.43	59.62	x_2=571.43; x_4=85.71
2	3200.00	8800.00	8400.00	14.33	x_3=800.0; x_4=40.0
3	0.00	0.00	15000.00	6.77	x_1=1500.0; x_6=2500.0
4	5333.33	1333.33	14000.00	19.28	x_1=1333.33; x_4=66.67

The characteristics of the basic efficient solutions to this problem are shown in table 1. x_5 and x_6 are the slacks variables of constraints ($\mathbf{c_1}$) and ($\mathbf{c_2}$), respectively. Fig. 1 displays the indifference regions associated with each basic efficient solution. Area denotes the percentage of the parametric diagram occupied by the indifference region.

Efficient solutions 1, 2 and 3 individually optimize objective functions z_1, z_2 and z_3, respectively. Solution 4 is the one studied in Hansen et al. [3] obtained with $\hat{\boldsymbol{\lambda}}$ =(0.1; 0.3; 0.6).

If the estimated weights $\hat{\boldsymbol{\lambda}}$ =(0.1; 0.3; 0.6) are considered and no a-priori information on the variation of weights exists, then the maximum tolerance value determined by the visual interactive tolerance approach proposed in [2] is 25%. The vertex of the convex polygon that originates this value lies on the intersecting line of indifference regions associated with solutions 3 and 4 in Fig. 2, denoted by an arrow. This means that any variation of up to 25% in the estimated values of those weights does not change the efficient solution; that is, $\hat{\lambda}_1$ can change in the range [0.075, 0.125], $\hat{\lambda}_2$ in [0.225, 0.375], and $\hat{\lambda}_3$ in [0.45, 0.75].

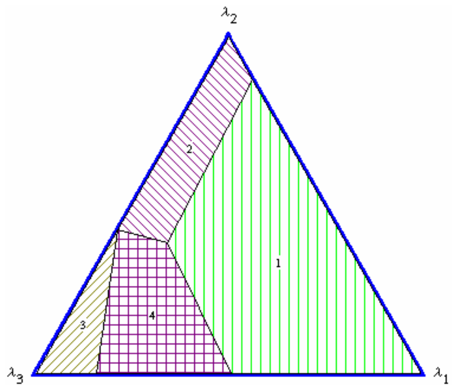

Figure 1: Indifference regions.

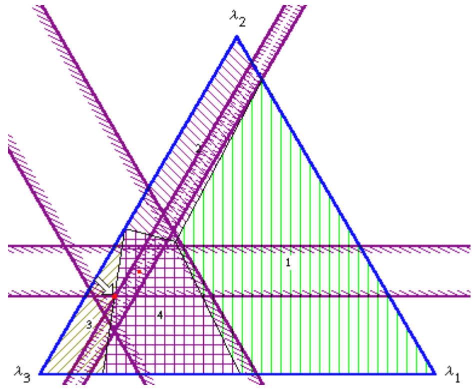

Figure 2: $\hat{\lambda} = (0.1; 0.3; 0.6)$.

Let us suppose that the centroid of the indifference region associated with solution 4, $\hat{\lambda}$ =(0.2207; 0.1720; 0.6073), is considered as the estimated weights. As it can be concluded by analyzing Fig. 3, the maximum tolerance value determined is 33.5252%, and the vertex of the convex polygon is now on the intersecting line of indifference regions associated with solutions 1 and 4 (see the arrow).

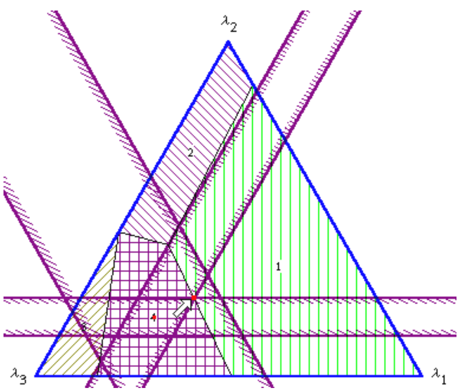

Figure 3: $\hat{\lambda} = (0.2207; 0.1720; 0.6073)$.

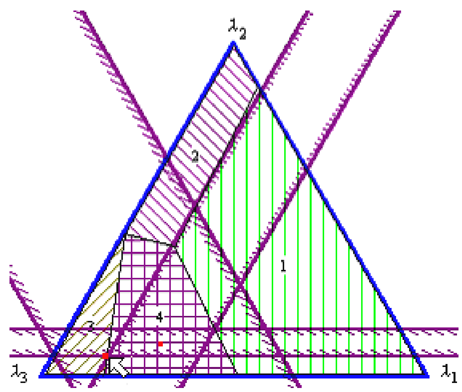

Figure 4: $\hat{\lambda} = (0.2607; 0.0920; 0.6473)$.

Fig. 4 shows the situation, without any a-priori information on the weights, and the selected weights $\hat{\lambda}$ =(0.2607; 0.0920; 0.6473). This is also leading to basic efficient solution 4, for which the maximum tolerance value is 46.4018%.

The maximum tolerance value computed (46.4018%) is higher than the one obtained under similar conditions on Fig. 2 (25%) and Fig. 3 (33.5252%). This means that the last set of weights analyzed tends to be more stable with respect to simultaneous and independent changes in weight values, than the ones previously studied.

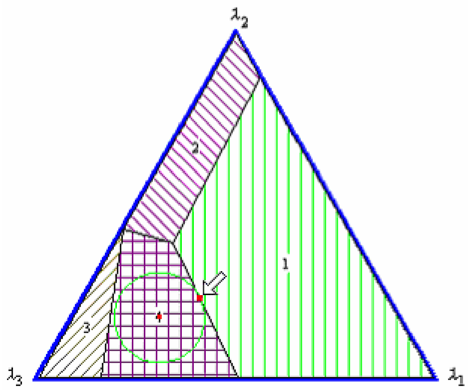

Figure 5: $\hat{\lambda}$ = (0.2207; 0.1720; 0.6073).

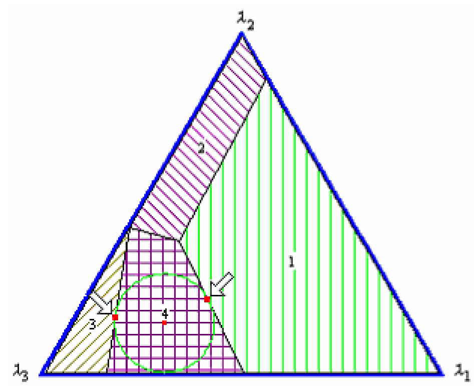

Figure 6: $\hat{\lambda}$ = (0.2318; 0.1456; 0.6225).

Although the set of weights corresponding to the centroid of the indifference region associated with an efficient solution is generally considered as the most robust against changes in the weights values, these experiments contradict this assertion.

Figs. 5 and 6 illustrate the use of the Euclidean metric to evaluate the stability of the weighting vector in the scope of the tolerance approach. Fig. 5 shows the largest circle (on the plane $\lambda_1+\lambda_2+\lambda_3=1$) centred on the weights corresponding to the centroid of the indifference region associated with solution 4. Fig. 6 shows the largest circle (on the plane $\lambda_1+\lambda_2+\lambda_3=1$) inside the indifference region associated with solution 4. Note that the circle in Fig. 6 is larger than the one in Fig. 5; therefore, the weights corresponding to the centroid of the indifference region associated with an efficient solution are not the more stable regarding simultaneous and independent changes of the weight values.

References

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