# Tolerance and critical regions of reference points: a study of bi-objective linear programming models

**Ana Rosa Borges**
ISEC, Coimbra Polytechnic Institute,
Rua Pedro Nunes, Quinta de Nora, 3030-199
and
INESC - Rua Antero de Quental 199, 3000-033
Coimbra, Portugal
arborges@isec.pt

**Carlos Henggeler Antunes**
Dep. of Electrical Engineering and Computers,
University of Coimbra, 3030-030
and
INESC - Rua Antero de Quental 199, 3000-033
Coimbra, Portugal
cantunes@inescc.pt

## Abstract

An interactive approach based on the analysis of the tolerance and critical regions of reference points associated with the objective functions in multiple objective linear programming models is proposed. It allows characterizing the efficient solution set as well as studying the stability of basic efficient solutions in presence of changes in the reference point components, capturing the uncertainty regarding the decision maker's preferences.

Some methodological and theoretical issues are presented and a bi-objective example is used for illustration purposes.

**Keywords:** Multiobjective linear programming, Sensitivity analysis, Tolerance approach, Reference point approaches, Interactive decision analysis.

## 1 Introduction

Generally, in linear programming, traditional sensitivity analysis ranges indicate how much given coefficients can change before the optimal basis changes. However, these ranges are easily computed only when the coefficients are not allowed to change in a simultaneous manner.

Wendell [6, 7] developed the tolerance approach to sensitivity analysis, which enables to consider the impact of simultaneous and independent coefficient changes for single objective linear programming problems. The tolerance approach proposed by Wendell yields the largest percentage, called the maximum tolerance percentage, by which specific coefficients can deviate simultaneously and independently from their estimated values while retaining the same optimal basic solution.

Some authors [3, 4] have proposed extensions of the tolerance approach to sensitivity analysis in linear programming to problems with multiple objectives, where efficient solutions are computed by using the weighted-sum approach. They consider perturbations on the weights of the p objective functions and determine the largest percentage, called the maximum tolerance percentage, by which all weights can deviate simultaneously and independently from their estimated values while retaining the same efficient basic solution.

Based on the geometrical analysis of the critical region (derived from traditional sensitivity analysis) and tolerance region for the weights, Borges and Antunes [1] presented a geometrical-based approach to determine the results in [3], along with visual interactive tools to deal with changes in the weighting vector in multiobjective linear programming (MOLP) problems.

Since reference point-based approaches are also generally used to compute efficient solutions, it is useful to integrate the main concepts underlying this form of scalarizing problems with the tolerance approach to sensitivity analysis in MOLP problems [2].

In this work, we endeavour to establish bridges between the tolerance approach to sensitivity analysis with the reference point-based scalarizing process as a means to deal with uncertainty in MOLP problems. The paper is organized as follows. In sections 2 and 3 the MOLP problem and the reference point scalarizing process are formulated, respectively. In section 4, the integration of the tolerance approach to sensitivity analysis with reference point approaches is described. The interactive approach is detailed in section 5. It allows characterizing the efficient solution set as well as providing support to the decision maker (DM) to study the stability of basic efficient solutions in presence of changes on reference point components. An illustrative example is presented in section 6. Finally, some conclusions are drawn in section 7.

## 2 Multiobjective linear programming

Let us consider the following MOLP problem with p linear objective functions and m linear constraints[1]

$$\text{“max”} \ \mathbf{z} = (z_1, ..., z_r, ..., z_p)^T = C\mathbf{x} \qquad (1)$$

s. t.

$$\mathbf{x} \in X = \left\{ \mathbf{x} \in \Re^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0 \right\}$$

where A is a mxn matrix, **b** is the m right-hand side column vector and C is a pxn matrix of objective functions coefficients.

Since it is usually impossible to optimize all the objectives simultaneously, the concept of optimal solution to a single objective problem gives place, in a multiple objective context, to the concept of efficient solutions. “max” denotes the operation of computing efficient solutions. A feasible solution to (1) is called efficient if and only if there is no feasible solution for which an improvement in any objective functions value is possible without sacrificing on at least one of the other objective functions. For definitions and mathematical details see, for instance, [5].

A variety of approaches have been developed to determine efficient solutions in MOLP problems. These processes are generally called scalarizing processes because they involve the resolution of a scalar optimization problem in such a way that the optimal solution to this problem is an efficient solution to the multiobjective problem. Scalarizing processes include, among others, the optimization of a non-negative weighted-sum of the objective functions and the use of achievement functions associated with reference points. The components of the reference point, in the objective function space, may be understood as the levels the DM would like to attain regarding each objective function (also called aspiration levels).

## 3 Reference point approaches in multiobjective linear programming

Reference point-based approaches [8, 9] provide an appealing framework, both from theoretical and cognitive perspectives, to aid the DM to strive for "satisfactory" efficient solutions.

Let $\mathbf{q} = (q_1, ..., q_r, ..., q_p)^T \in \Re^p$ be the reference point in the objective space. It may be attainable or not.

An achievement scalarizing function, $\sigma(\mathbf{q}, \mathbf{z})$, is used to project the reference point **q** on the efficient region to determine an efficient solution.

Although several, more generalized, forms of this function have been presented in the literature [8, 9] a simplified version of it is used in this work

$$\sigma(\mathbf{q}, \mathbf{z}) = \max_{1 \leq r \leq p} \left\{ (q_r - z_r) + \varepsilon \sum_{r=1}^{p} (q_r - z_r) \right\} \quad (2)$$

where the parameter $\varepsilon$ is an arbitrary small positive number; if $\varepsilon$ is zero, alternative optima to this function may lead to weakly efficient solutions.

[1] Without loss of generality, it is assumed that the objective functions are given in maximization form and all constraints are equalities.

The reference point problem can be formulated as

$$\max\ \alpha + \varepsilon \textstyle\sum_{r=1}^{p} z_r \qquad (3)$$

s. t.

$$z_r - \alpha \geq q_r\,; \qquad r = 1,\ldots,p$$
$$\mathbf{x} \in X\,,\ \alpha \in \Re\,.$$

For a given $\mathbf{q} \in \Re^p$, the optimal solution to problem (3) is an efficient solution to problem (1).

# 4 Geometrical interpretation of the tolerance and critical regions of reference points

## 4.1 The problem

The tolerance approach to sensitivity analysis of the reference point values in (3) is developed through the following perturbed problem [2]

$$\max\ \alpha + \varepsilon \textstyle\sum_{r=1}^{p} z_r \qquad (4)$$

s. t.

$$z_r - \alpha \geq \hat{q}_r + \delta_r\, q'_r\,; \qquad r = 1,\ldots,p$$
$$\mathbf{x} \in X\,,\ \alpha \in \Re\,.$$

where each $q'_r$ has a specified value and $\delta_r$ is the multiplicative parameter of $q'_r$. More specifically, if $q'_r = \hat{q}_r$ then $\delta_r$ represents the percentage deviation from the estimated value $\hat{q}_r$.

## 4.2 Tolerance region of reference points

The tolerance region can be determined graphically from a set of hypercubes centered on the estimated values (the reference point) and with radius equal to the tolerance of the corresponding component. The maximum tolerance value is associated with the largest hypercube for which all points in the hypercube represent changes to the reference values such that the efficient basis obtained with the estimated reference values remains efficient. This hypercube represents the tolerance region for these reference values.

In the example with two objective functions (see Figure 1) the hypercubes are rectangles. The tolerance region for the reference values is the dashed rectangle because it is the smallest of all (two) rectangles.

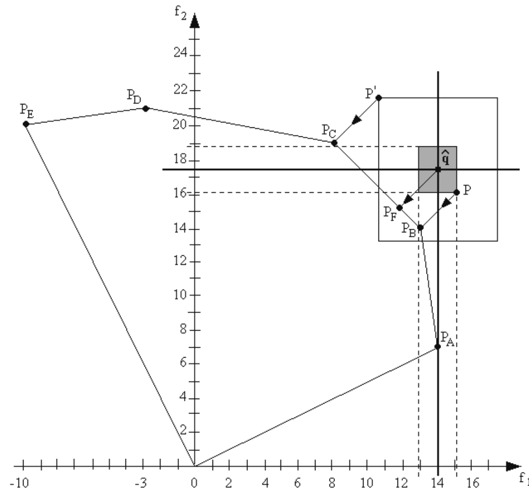

Figure 1: Tolerance region obtained from Wendell's approach with $\hat{\mathbf{q}}$=[14.0, 17.5].

Note that this tolerance region lies inside the critical region of the objective function reference point components, obtained by a traditional sensitivity analysis study for a certain optimal basic solution to problem (3) (using $\hat{\mathbf{q}}$).

## 4.3 Critical region of reference points

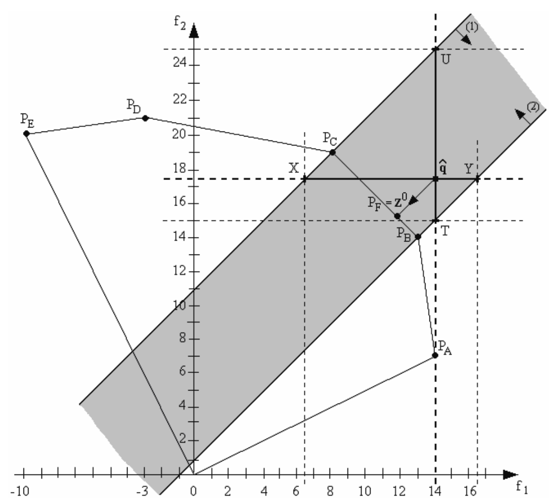

Figure 2: Critical region with $\hat{\mathbf{q}} = \mathbf{q}^0$ =[14.0, 17.5].

From a geometrical analysis of figure 2, it can be concluded that if we project any point belonging to the line segment [T, U] (with T=(14.0, 15,0) and U=(14.0, 25,0)), as well as any point belonging to the line segment [X, Y] (with X=(6.5, 17,5) and Y=(16.5, 17,5)), on the efficient region boundary, a solution that can be determined by a linear combination of basic efficient solutions $P_B$ and $P_C$ is obtained (for instance, the efficient solution $P_F$).

Note that the values considered are related with the ranges computed by traditional sensitivity analysis, $[q_1^{min}, q_1^{max}]$ = [6.5, 16.5] and $[q_2^{min}, q_2^{max}]$ = [15.0, 25,0]. Moreover, the critical region for the reference point components can be found by an analysis based on these ranges.

Furthermore, for the illustrative problem in figure 2, this critical region could be defined by means of the vectors that are projected on the points $P_C$ and $P_B$, respectively. That is, all the points on parallel straight lines to the vectors $\overrightarrow{XU}$ (or $\overrightarrow{TY}$) and which intersect points of the line [X, Y] (or [T, U]) belong to the corresponding critical region.

Equations of both half-planes (1) and (2) in figure 2, that bound the critical region, can be defined by:

$$(q_2 - q_2^{max})(q_1^{min} - \hat{q}_1) \geq (\hat{q}_2 - q_2^{max})(q_1 - \hat{q}_1) \quad (5.(1))$$

$$(q_2 - q_2^{min})(q_1^{max} - \hat{q}_1) \geq (q_1 - \hat{q}_1)(\hat{q}_2 - q_2^{min}) \quad (5.(2))$$

For a given optimal basis B of problem (3), the critical region of reference point components could also be characterized by the following set:

$$R_{\mathbf{q}} = \{\mathbf{q}: B^{-1}[\mathbf{q}\ \mathbf{b}] \geq 0\} \quad (6)$$

In linear problems, the critical region of reference point is a polytope and the maximal tolerance region lies always inside the critical region.

A different way to characterize the critical region is suggested from figure 2, where X=$(q_1^{min}, \hat{q}_2)$, U=$(\hat{q}_1, q_2^{max})$, Y=$(q_1^{max}, \hat{q}_2)$ and T=$(\hat{q}_1, q_2^{min})$. Let us begin to analyze the half-plane (1).

Each vector (in the space) beginning at point U and ending at any point q=$(q_1, q_2, 0)$ of the corresponding half-plane (1), $\overrightarrow{Uq} = (q_1 - \hat{q}_1, q_2 - q_2^{max}, 0)$, is at an angle between 0º and 180º with the vector $\overrightarrow{XU} = (\hat{q}_1 - q_1^{min}, q_2^{max} - \hat{q}_2, 0)$. Each vector (in the space) beginning at point U and ending at any point q=$(q_1, q_2, 0)$, of the corresponding half-plane (1), $\overrightarrow{Uq} = (q_1 - \hat{q}_1, q_2 - q_2^{max}, 0)$, is at an angle between 0º and 180º with the vector $\overrightarrow{XU} = (\hat{q}_1 - q_1^{min}, q_2^{max} - \hat{q}_2, 0)$.

Let $\overrightarrow{e_{f3}}$=(0, 0, 1) be an unit vector perpendicular to the plane to which $\overrightarrow{XU}$ and $\overrightarrow{Uq}$ belong, and such that $(\overrightarrow{e_{f3}}, \overrightarrow{Uq}, \overrightarrow{XU})$ form a right-handed system in $\Re^3$ (direct base on the space). The triple product[2] $\overrightarrow{e_{f3}} \bullet (\overrightarrow{Uq} \times \overrightarrow{XU}) \geq 0$.

That is,

$$(0, 0, 1) \bullet \quad (7.(1))$$

$$(0, 0, (q_1 - \hat{q}_1)(q_2^{max} - \hat{q}_2) - (q_2 - q_2^{max})(\hat{q}_1 - q_1^{min})) \geq 0$$

$$(q_1 - \hat{q}_1)(q_2^{max} - \hat{q}_2) - (q_2 - q_2^{max})(\hat{q}_1 - q_1^{min}) \geq 0$$

$$(q_1 - \hat{q}_1)(q_2^{max} - \hat{q}_2) \geq (q_2 - q_2^{max})(\hat{q}_1 - q_1^{min})$$

$$(q_1 - \hat{q}_1)(\hat{q}_2 - q_2^{max}) \leq (q_2 - q_2^{max})(q_1^{min} - \hat{q}_1)$$

The last equation is the previous (5.(1)).

A similar procedure could be drawn to half-plane (2), and the constraint (5.(2)) is obtained.

Each vector beginning at point T and ending at any point q=$(q_1, q_2, 0)$ of the half-plane (2), $\overrightarrow{Tq} = (q_1 - \hat{q}_1, q_2 - q_2^{min}, 0)$, is at an angle between 0º and 180º with the vector $\overrightarrow{TY}$ = $(q_1^{max} - \hat{q}_1, \hat{q}_2 - q_2^{min}, 0)$.

---

[2] The triple product absolute value of three vectors is equal to the volume of the parallelepiped spanned by those vectors. When the three vectors are complanar the value is null. The sign is positive (≥ 0) if the vectors form a right-handed system, and in this situation the parallelepiped defined by the three vectors is completely located at the half-space to which the vectors belong. The sign of the triple product is negative otherwise.

Let $\overrightarrow{e'_{f3}}$ =(0, 0, -1) be an unit vector perpendicular to the plane to which $\overrightarrow{Tq}$ and $\overrightarrow{TY}$ belong, and such that ($\overrightarrow{e'_{f3}}$, $\overrightarrow{Tq}$, $\overrightarrow{TY}$) is a right-handed system in $\Re^3$. Then, the triple product $\overrightarrow{e'_{f3}} \cdot (\overrightarrow{Tq} \times \overrightarrow{TY}) \geq 0$, that is:

$$(0, 0, -1) \cdot \qquad (7.(2))$$

$$(0,0,(q_1 - \hat{q}_1)(\hat{q}_2 - q_2^{min}) - (q_2 - q_2^{min})(q_1^{max} - \hat{q}_1)) \geq 0$$

$$-(q_1 - \hat{q}_1)(\hat{q}_2 - q_2^{min}) + (q_2 - q_2^{min})(q_1^{max} - \hat{q}_1) \geq 0$$

$$(q_1 - \hat{q}_1)(\hat{q}_2 - q_2^{min}) \leq (q_2 - q_2^{min})(q_1^{max} - \hat{q}_1).$$

## 5 An interactive approach based on the analysis of the critical regions of reference points

The proposed interactive approach to deal with uncertainty with respect to the reference point components in MOLP problems consists of the following steps:

1. The solutions that optimize each objective function individually are computed. This offers the DM a first overview of the ranges attained by the objective functions in the efficient region. If the DM is also interested in knowing more basic efficient solutions with different characteristics, they could be determined at this phase.
2. The DM is asked to specify the objective function reference levels he would like to attain: $\mathbf{q}^0$ (iteration t=0). If the DM is not willing to specify some of these levels, those that optimize each objective function are considered.
3. Problem (3) is solved in order to compute an efficient solution to the multiobjective problem (1).
4. The results obtained are presented to the DM. The ranges in which each reference value can change independently before the efficient basis changes, obtained by traditional sensitivity analysis, as well as the maximum tolerance percentage (that gives the deviation the reference values may suffer simultaneously and independently from their estimated values while retaining the same efficient basis), are also provided.
5. If the DM is satisfied with the results obtained then the interactive procedure can be concluded. Otherwise, the DM is asked to specify a different $\mathbf{q}^{t+1}$. This aims at capturing the uncertainty regarding his preferences before more information is gathered about the efficient solutions set.

   If, with respect to a previous $\mathbf{q}^t$ obtained, the critical region of the reference levels remain the same, then the new efficient solution can be easily determined by computing $\mathbf{x}_B^{t+1} = B^{-1}\left[\mathbf{b}\ \mathbf{q}^t\right]^T$ and $\mathbf{z}^{t+1} = C_B B^{-1}\left[\mathbf{b}\ \mathbf{q}^t\right]^T$, with the appropriate $B^{-1}$ and $C_B$, and the interactive process proceeds to step 4. Otherwise, it is necessary to return to step 3 and solve the corresponding problem (3) from scratch.

## 6 Some illustrative results

To illustrate the proposed approach, let us consider the following MOLP problem with three objective functions (already used in section 4, Figures 1 and 2).

$$\text{“max”}\ \mathbf{z} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -1x_1 & +2x_2 \\ 2x_1 & +1x_2 \end{pmatrix} \qquad (8)$$

s. t.

$$X = \begin{cases} -1x_1 + 3x_2 \leq 21 & \text{(c1)} \\ 1x_1 + 3x_2 \leq 27 & \text{(c2)} \\ 4x_1 + 3x_2 \leq 45 & \text{(c3)} \\ 3x_1 + 1x_2 \leq 30 & \text{(c4)} \\ \mathbf{x} \in \Re^2, \mathbf{x} \geq 0 \end{cases}$$

Firstly, the characteristics of the basic efficient solutions that optimize each objective function individually are presented to the DM (table 1). sl_ci is the slack variable associated with the constraint (ci).

The DM may also find interesting to compute other basic efficient solutions to the problem, with different characteristics. For instance, by using the weighted-sum approach, the basic efficient solutions presented in table 2 can also be determined.

Table 1: Initial basic efficient solutions to problem (8)

| **Solution** | $f_1$ | $f_2$ | $\mathbf{x}_B$ |
|---|---|---|---|
| $P_A$ (max $f_1$) | 14.00 | 7.00 | $(x_2$, s_c2, s_c3, s_c4) = (7.0, 6.0, 24.0 ,23.0) |
| $P_D$ (max $f_2$) | -3.00 | 21.00 | $(x_1, x_2$, s_c1, s_c2) = (9.0, 3.0, 21.0 ,9.0) |

Table 2: Other basic efficient solutions to problem (8)

| **Solution** | $f_1$ | $f_2$ | $\mathbf{x}_B$ |
|---|---|---|---|
| $P_B$ | 13.00 | 14.00 | $(x_1, x_2$, s_c3, s_c4) = (3.0, 8.0, 9.0 ,13.0) |
| $P_C$ | 8.00 | 19.00 | $(x_1, x_2$, s_c1, s_c4) = (6.0, 7.0, 6.0 ,5.0) |

Let us suppose that the DM specifies the reference point $\mathbf{q}^0$ =[14.0, 17.5]$^T$ (Figure 2).

The following problem, similar to problem (3), is then solved:

$$\max \quad \alpha^+ - \alpha^- + \varepsilon\ (x_1 + 3x_2) \tag{9}$$

s. t.

$$-1x_1 + 2x_2 - \alpha^+ + \alpha^- \geq 14.0 \qquad \text{(c5)}$$
$$2x_1 + 1x_2 - \alpha^+ + \alpha^- \geq 17.5 \qquad \text{(c6)}$$

$$\mathbf{x} \in X, \quad \alpha^+ \geq 0, \; \alpha^- \geq 0 .$$

The optimal values to the problem (9) are presented in Table 3, where $\alpha = \alpha^+ - \alpha^-$.

$\mathbf{x}_B{}^0$ = $[x_1, \alpha^-$, s_c3, s_c4, $x_2$, s_c1$]^T$ = [3.75, 2.25, 6.75, 11, 7.75, 1.5]$^T$, the objective values are $\mathbf{z}^0$=[11.75, 15.25]$^T$ (point $P_F$) and the inverse of the optimal basis matrix to the problem (9) is

$$B^{-1} = \begin{bmatrix} 0 & 1/10 & 0 & 0 & -3/10 & 3/10 \\ 0 & -1/2 & 0 & 0 & 1/2 & 1/2 \\ 0 & -13/10 & 1 & 0 & 9/10 & -9/10 \\ 0 & -3/5 & 0 & 1 & 4/5 & -4/5 \\ 0 & 3/10 & 0 & 0 & 1/10 & -1/10 \\ 1 & -4/5 & 0 & 0 & -3/5 & 3/5 \end{bmatrix}.$$

Table 3: Original and optimal tableau to problem (9)

| $\mathbf{x}_B$ | $x_1$ | $x_2$ | $\alpha^+$ | $\alpha^-$ | s_c1 | s_c2 | s_c3 | s_c4 | s_c5 | s_c6 | Current variables |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Original tableau | | | | | | | | | | | |
| s_c1 | -1 | 3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 21.00 |
| s_c2 | 1 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 27.00 |
| s_c3 | 4 | 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 45.00 |
| s_c4 | 5 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 30.00 |
| s_c5 | -1 | 2 | -1 | 1 | 0 | 0 | 0 | 0 | -1 | 0 | 14.00 |
| s_c6 | 2 | 1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 17.50 |
| Optimal tableau | | | | | | | | | | | |
| $x_1$ | 1 | 0 | 0 | 0 | 0 | 1/10 | 0 | 0 | 3/10 | -3/10 | 3.75 |
| $\alpha^-$ | 0 | 0 | -1 | 1 | 0 | -1/2 | 0 | 0 | -1/2 | -1/2 | 2.25 |
| s_c3 | 0 | 0 | 0 | 0 | 0 | -13/10 | 1 | 0 | -9/10 | 9/10 | 6.75 |
| s_c4 | 0 | 0 | 0 | 0 | 0 | -3/5 | 0 | 1 | -8/10 | 8/10 | 11 |
| $x_2$ | 0 | 1 | 0 | 0 | 0 | 3/10 | 0 | 0 | -1/10 | 1/10 | 7.75 |
| s_c1 | 0 | 0 | 0 | 0 | 1 | -4/5 | 0 | 0 | 3/5 | -3/5 | 1.50 |

The intervals obtained from traditional sensitivity analysis for each reference value can also be provided[3]: [6.5; 16.5] for $f_1$ and [15; 25] for $f_2$, as well as the maximum tolerance percentage[4] (that is, the same basis remains efficient as long as each of the reference values of the objective functions is within 7.9365% of its estimated value $\mathbf{q}^0 = [14.0, 17.5]^T$).

The analysis of Figures 1 and 2 leads to similar conclusions. The critical region of the estimated reference point $\mathbf{q}^0$, shown in figure 2, can be defined through equation (6):

$$R_{\mathbf{q}} = \left\{ \mathbf{q} : \begin{bmatrix} 0 & 1/10 & 0 & 0 & -3/10 & 3/10 \\ 0 & -1/2 & 0 & 0 & 1/2 & 1/2 \\ 0 & -13/10 & 1 & 0 & 9/10 & -9/10 \\ 0 & -3/5 & 0 & 1 & 4/5 & -4/5 \\ 0 & 3/10 & 0 & 0 & 1/10 & -1/10 \\ 1 & -4/5 & 0 & 0 & -3/5 & 3/5 \end{bmatrix} \begin{bmatrix} 21 \\ 27 \\ 45 \\ 30 \\ q_1 \\ q_2 \end{bmatrix} \geq \mathbf{0} \right\}$$

that is, it can be characterized by the intersection of the constraints:

$$27/10 - 3/10q_1 + 3/10q_2 \geq 0 \qquad (\mathbf{r}1)$$
$$\Leftrightarrow q_1 - q_2 \leq 9$$

$$-27/2 + 1/2q1 + 1/2q2 \geq 0, \qquad (\mathbf{r}2)$$

$$-13*27/10 + 45 + 9/10q_1 - 9/10q_2 \geq 0 \qquad (\mathbf{r}3)$$
$$\Leftrightarrow q_1 - q_2 \geq -11$$

$$-3*27/5 + 30 + 8/10q_1 - 8/10q_2 \geq 0 \qquad (\mathbf{r}4)$$
$$\Leftrightarrow q_1 - q_2 \geq -69/4$$

$$3*27/10 + 1/10q_1 - 1/10q_2 \geq 0 \qquad (\mathbf{r}5)$$
$$\Leftrightarrow q_1 - q_2 \geq -81$$

$$21 - 4*27/5 - 3/5q_1 + 3/5q_2 \geq 0 \qquad (\mathbf{r}6)$$
$$\Leftrightarrow q_1 - q_2 \leq -1$$

Since the constraints (**r**1), (**r**4) and (**r**5) are redundant, and (**r**2) must not be considered (owing to the fact that it is associated with the basic variable $\alpha^-$), only equations (**r**3) and (**r**6) remain. Those equations are similar to equation determined by (5.(1)) e de (5.(2)), respectively:

$$q_1 - q_2 \geq -11 \qquad (\mathbf{r}3)$$
$$q_1 - q_2 \leq -1 \qquad (\mathbf{r}6)$$

Let us suppose that, based on information already gathered, the DM specifies a new reference point, for instance the dominated and attainable point $\hat{\mathbf{q}} = \mathbf{q}^1 = [7.0, 11]^T$ (Figure 3).

Since the critical region of reference points ($\mathbf{q}^0$ and $\mathbf{q}^1$) does not change, the new solution can be easily determined by $\mathbf{x}_B^1 = B^{-1}\left[\mathbf{b}\ \mathbf{q}^1\right]^T = [x_1, \alpha^-, s\_c3, s\_c4, x_2, s\_c1]^T = [3.9, -4.5, 6.3, 10.6, 7.7, 1.8]^T$ and $\mathbf{z}^1 = C_B B^{-1}\left[\mathbf{b}\ \mathbf{q}^1\right]^T = [11.5, 15.5]^T$.

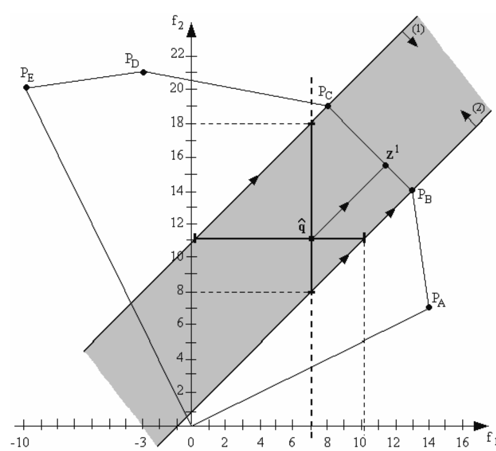


Figure 3: Critical region with $\hat{\mathbf{q}} = \mathbf{q}^1 = [7.0, 11.0]$.

The interactive process proceeds to step 4.

If the DM specify new values $\mathbf{q}^2 = [16.67, 11.0]$ to the reference point components, the critical region changes (Figure 4). This means that the

---

[3] $q_1^{min} = q_1 + \max\{-6.75/(9/10), -11/(8/10), -7.75/(1/10)\} = 6.5$;

$q_1^{max} = q_1 + \min\{-3.75/(-3/10), -1.5/(-3/5)\} = 16.5$;

$q_2^{min} = q_2 + \max\{-3.75/(3/10), -1.5/(3/5)\} = 15$;

$q_2^{max} = q_2 + \min\{-6.75/(-9/10)\}, -11/(-8/10), -7.75/(-1/10)\} = 25$.

[4] min{3.75/(|-3/10|*14 + |3/10|*17.5), 6.75/(|9/10|*14 + |-9/10|*17.5), 11/(|8/10|*14 + |-8/10|*17.5), 7.75/(|1/10|*14 + |-1/10|*17.5), 1.5/(|-3/5|*14 + |3/5|*17.5) } = 1.5/(|-3/5|*14 + |3/5|*17.5)=7.9365%.

Note that α is always negative because 2.25/(|1/2|*14 + |1/2|*17.5) = 14.2857% is greater than the maximum tolerance percentage value, 7.9365%.

new efficient solution could now be obtained by using a linear combination of different basic efficient solutions ($P_B$ and $P_A$).

$\mathbf{x}_B^{2} = B^{-1}\left[\mathbf{b}\ \mathbf{q}^2\right]^T$ = [ $x_1$, $\alpha^-$, $x_2$, s_c2, s_c3, s_c4]$^T$ = [0.5, 2.8, 7.1667, 5, 21.5, 21.3333]$^T$ and $\mathbf{z}^2 = C_B B^{-1}\left[\mathbf{b}\ \mathbf{q}^2\right]^T$ =[13.8333, 8.1667]$^T$.

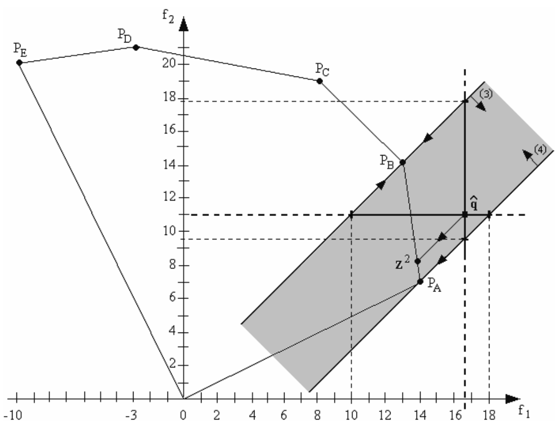


Figure 4: Critical region with $\hat{\mathbf{q}} = \mathbf{q}^2$ =[16.67, 11.0].

If the DM considers the information obtained sufficient to make a decision, the process can successfully be concluded. Otherwise, new reference levels must be specified to proceed with a new interaction. This offers the DM a way to progressively reduce the uncertainty associated with his preferences as more knowledge about the efficient solutions set is gathered.

## 7 Conclusions

The main objective of this work is to propose an interactive approach, which does not impose a heavy computer effort, to aid the DM in the study of the efficient solution set to a MOLP problem. This approach provides an operational means to analyze the stability of basic efficient solutions regarding changes in the reference point components which capture the DM's uncertainty in specifying his own preferences.


## Acknowledgements

This research has been partially supported by FCT and FEDER under project grant POSI/SRI/37346/2001.


## References


[1] Borges, A.R., C.H. Antunes (2002). A visual interactive tolerance approach to sensitivity analysis in MOLP. European Journal of Operational Research, Vol. 142, nº 2, 357-381.

[2] Borges, A.R., C.H. Antunes (2002). An interactive tolerance approach to sensitivity analysis in multiobjective linear programming using reference points. In Proceedings of the 2002 IEEE International Conference on Systems, Man and Cybernetics (SMC'02), A. El Kamel, K. Mellouli, P. Borne (Eds.), Hammamet, Tunisia, (CD-ROM).

[3] Hansen, P., M. Labbé, R.E. Wendell (1989). Sensitivity analysis in multiple objective linear programming: The tolerance approach, European Journal of Operational Research 38, 63-69.

[4] Mármol, A., J. Puerto (1997). Special cases of the tolerance approach in multiobjective linear programming, European Journal of Operational Research 98, 610-616.

[5] Steuer, R. (1986). Multiple Criteria Optimization: Theory, computation and Application, Wiley, New York.

[6] Wendell, R.E. (1984). Using bounds on the data in linear programming: the tolerance approach to sensitivity analysis, Mathematical Programming 29, 304-322.

[7] Wendell, R.E. (1985). The tolerance approach to sensitivity analysis in linear programming, Management Science 31, 564-578.

[8] Wierzbicki, A.P. (1983). A mathematical basis for satisficing decision making. Mathematical Modeling, 3, 391-405.

[9] Wierzbicki, A.P. (2000). Multi-objective and reference point optimization tools. In Model-based decision support methods with environmental applications (Wierzbicki, A.P., M. Makowski and J. Wessels (Ed)), Kluwer Academic Publishers, Dordrecht, 215-247.