Robustness Analysis in Evolutionary Multi-Objective Optimization

Carlos Barrico[1,2] and Carlos Henggeler Antunes[1,3]

[1] INESC Coimbra, Rua Antero de Quental 199, 3000-033 Coimbra, Portugal.
{cbarrico, cantunes}@inescc.pt

[2] Department of Informatics, University of Beira Interior, 6201-001 Covilhã, Portugal.
cbarrico@di.ubi.pt

[3] Department of Electrical Engineering and Computers, University of Coimbra, Portugal.
ch@deec.uc.pt

Contents

Abstract

This paper presents two approaches to robustness analysis in multi-objective optimization problems, in which the model data (coefficients of objective functions, coefficients of constraints, bounds of decision variables, etc.) are subject to small perturbations, with respect to a "nominal" set of coefficients of the model data. In these approaches, the concept of degree of robustness is incorporated into an evolutionary algorithm, being operationalized in the computation of the fitness value assigned to solutions. Non-dominated solutions are classified according to their degree of robustness. The information on the degree of robustness of solutions is provided to support the decision maker in the selection of a robust compromise solution.

1 Introduction

Most complex problems arising in modern technologically developed societies inherently involve multiple, conflicting, and incommensurate evaluation aspects to assess the merits of alternative courses of action. Therefore, mathematical models for decision support become more representative of the actual decision context if those distinct axes of evaluation are explicitly taken into account rather than aggregated into a single indicator, generally of economic nature such as cost or benefit. In multi-objective programming models those axes of evaluation are operationalized by means of objective functions to be optimized subject to a set of constraints. Multi-objective models enable to grasp the conflicting nature of the objectives and the tradeoffs to be made in order to identify satisfactory compromise solutions by providing a basis to rationalize the comparison between non-dominated solutions. A non-dominated (efficient, Pareto optimal) solution is a feasible solution for which no improvement in all objective functions is simultaneously possible; that is, an improvement in an objective function can only be achieved by degrading, at least, another objective function value. Besides contributing to make the model more realistic vis-a-vis actual problems, a multi-objective approach intrinsically possesses a value-added role in the modeling process and in model analysis, supporting reflection and creativity of decision makers in face of a larger universe of potential solutions, having in mind their practical implementation, since a prominent solution no longer exists.

Uncertainty is an intrinsic characteristic of real-world problems arising from multiple sources of distinct nature. Uncertainty emerges from the ever-increasing complexity of interactions within social, economic and technical systems, characterized by a fast pace of technological evolution, changes in market structures, and new societal concerns. It is generally impracticable that decision aid models could capture all the relevant inter-related phenomena at stake, get through all the necessary information, and also account for the changes and/or hesitations associated with the expression of the stakeholders' preferences. Besides structural uncertainty associated with the global knowledge about the system being modeled, input data may also suffer from imprecision, incompleteness or be subject to changes. In this context, it is important to provide decision makers with robust conclusions. The concept of robust solution is generally linked to a certain degree of "immunity" to data uncertainty, to an adaptive capability (or flexibility) regarding an uncertain future or ill-specified preferences, guaranteeing an acceptable performance even under changing conditions (such as model coefficients drifting from "nominal data").

The ability to work in each generation with a population of potential solutions makes evolutionary approaches well suited for multi-objective optimization problems, particularly complex problems of combinatorial nature, in which a set of non-dominated solutions must be identified rather than a single optimal solution (see Coello et al. [8], Deb [10] and Fonseca and Fleming [14]).

The study of a multi-objective optimization problem generally involves the characterization of the set of non-dominated solutions, by performing either an exhaustive computation of these solutions or by computing a representative sample. However, some of these solutions, which could be of interest for a decision maker (DM) as acceptable compromise solutions, in the sense they present a satisfactory balance between the axes of evaluation operationalized through objective functions, can be very sensitive to perturbations. That is, when a given non-dominated solution, selected by using any approach, is implemented in practice, small variations regarding the values estimated for the model data (coefficients of objective functions, coefficients of constraints, bounds of decision variables, etc.) may lead to an important degrading of the objective function performances. Therefore, the need arises to offer the DM solutions that are relatively insensitive to perturbations of these parameters. That is, algorithms must strive for robust solutions.

Some studies have been devoted to compute robust solutions both in single-objective as well as in multi-objective evolutionary optimization.

The works dealing with robust single-objective evolutionary optimization include the following approaches. Branke [5] suggested some heuristics for computing robust solutions. Branke [6] pointed out key differences between searching for optimal solutions in a noisy environment and searching for robust solutions. Branke and Schmidh [7] suggested a number of methods for alternate fitness estimation. Jin and Sendhoff [18] considered an approach for finding robust solutions in single-objective optimization as a bi-objective optimization problem, in which the objectives to be maximized are the robustness and the performance related with the original function. Tsutsui and Ghosh [25] presented a mathematical model for obtaining robust solutions using the schema theorem for single-objective genetic algorithms. Parmee [22] suggested a hierarchical strategy for searching in several regions with high performance and in the fitness landscape simultaneously. Lim et al. [20] presented an evolutionary design optimization approach that handles uncertainty with respect to the desired robust performance (the so-called inverse robust design), from which they search for solutions that guarantee a certain degree of maximum uncertainty and, at the same time, satisfy the desired nominal performance of the final design solution. Ong et al. [21] presented an evolutionary algorithm based on the combination of a max-min optimization strategy with a Baldwinian trust-region framework, employing local surrogate models for reducing the computational cost associated with robust design problems (focusing on combining evolutionary algorithms with function approximation techniques for robust design).

As far as robust multi-objective optimization is concerned few studies are reported in the literature. Hughes [16] computed the expected error to be used in the deterministic Pareto dominance that depends on the noise in the objective functions. Teich [24] extended existing techniques for space exploration design based on the Pareto-dominance criterion to the case where one or more of the objective functions are subject to uncertainties given by property intervals. Li et al. [19] presented a Robust Multi-Objective Genetic Algorithm (RMOGA) to investigate the trade-off between the performance and the robustness of solutions, considering two objective functions, a fitness value and a robustness index. Deb and Gupta [11, 12] extended an existing approach for finding robust solutions in single-objective optimization problems to a multi-objective setting, by considering the mean effective objective functions computed by averaging a representative set of neighboring solutions instead of the original objective functions. Barrico and Antunes [3, 4] presented approaches that use the concept of degree of robustness, which are based on the solution behavior in its neighborhood in the decision variable space. The concept of degree of robustness is also used to assess the solution behavior in the neighborhood of the reference scenario in the space of the objective function coefficients (Barrico and Antunes [2]).

For more details about both robust single-objective and robust multi-objective evolutionary optimization see Jin and Branke [17].

In this paper, two evolutionary approaches to robustness analysis in multi-objective optimization are described, involving the definition of a degree of robustness (see Barrico and Antunes [2, 3, 4]). These approaches are aimed at problems subject to small perturbations occurring in the decision variable space and in the objective function coefficients. Both approaches are based on the solution behavior in its neighborhood when subject to those perturbations (see also the concept of robust solution of type II in Deb and Gupta [11, 12]). The concept of degree of robustness permits the user to exert a control on the desired level of robustness of the solutions obtained. Users can specify the size of the solution neighborhood, both in the space of the coefficients of the parameters subject to small perturbations (in the decision variable space or in the objective function coefficients) and in the objective function space (generally the space the DM is more familiar with).

The concept of degree of robustness is imbedded in the evolutionary process, particularly in the fitness assessment of each individual. The underlying rationale is to bias the evolutionary process towards more robust solutions, that is solutions for which the objective function performances are more immune to perturbations in the decision variable and coefficient values.

The goals of the evolutionary algorithm are finding a non-dominated front containing the more robust non-dominated solutions and also ensuring its diversity along the front.

In section 2 the concept of degree of robustness is presented. The main features of the evolutionary algorithm used for testing the approaches proposed are described in section 3. Illustrative results are presented in sections 4 and 5, which have been obtained with the evolutionary approach applied to bi-objective test problems, considering small perturbations in the decision variable space (section 4) and in the coefficients of objective functions (section 5). Finally, some conclusions are drawn in section 6.

2 The degree of robustness

The definition of robustness (either robust solution or robust method) is not uniform in the literature. However, a common view is shared: a robust solution shall behave well in (slightly) different conditions, meaning that it is as much as possible immune to small changes in the conditions it was designed for.

2.1 Perturbations in the decision variable space

For this case, it is assumed that perturbations of solution x may occur in any dimension $(x_1, x_2, ..., x_n)$, where n is the number of the decision variables. The assessment of the degree of robustness of a solution is based on its behavior within a neighborhood around its "nominal" point. The underlying idea is to determine a set of neighborhoods $k\delta$ around solution x, such that the images of solutions within these neighborhoods are better than $f(x)$ or belong to a pre-specified neighborhood η around $f(x)$ in the objective function space. The process begins by analyzing randomly generated solutions inside a hyperbox of radius δ around x. This neighborhood (hyperbox) is then progressively enlarged, in multiples of δ $(\delta, 2\delta, ...)$, until the percentage of solutions whose images in the objective space are better than $f(x)$ or belong to the η-neighborhood of $f(x)$ is lower than a pre-defined threshold. This enables to assign a degree of robustness to solutions according to the number of hyperbox enlargements for which that condition is fulfilled (see Fig. 1).

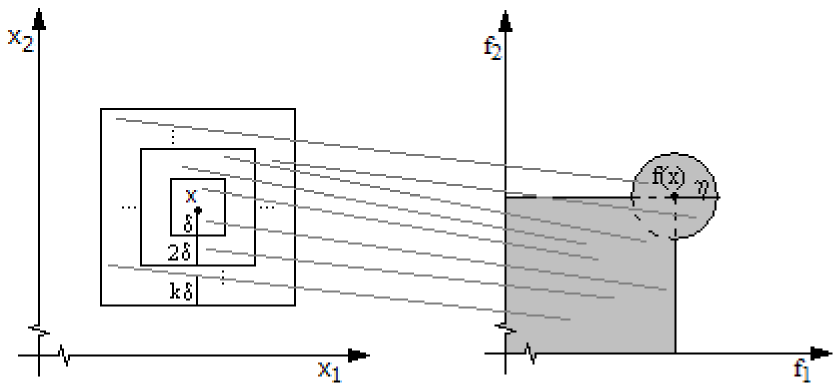

Figure 1: Definition of neighborhoods in the decision variable space and objective function space (for 2-dimension spaces and all functions to be minimized).

The degree of robustness depends on the size of a δ-neighborhood of solution x and the percentage of the h neighboring points whose objective function values are better than $f(x)$ or belong to the η-neighborhood of $f(x)$. Those h neighboring points are randomly generated around solution x (see also Deb and Gupta [11]).

The degree of robustness of solution x is a value k, such that (see Fig. 1):

a) the percentage of solutions in the $k\delta$-neighborhood of x, whose objective function values are better than $f(x)$ or belong to the η-neighborhood of $f(x)$, is greater than or equal to a pre-specified threshold p;

b) the percentage of solutions in the $(k+1)\delta$-neighborhood of x, whose objective function values are better than $f(x)$ or belong to the η-neighborhood of $f(x)$, is lower than p.

The degree of robustness k of a solution x is gradually computed as k increases (neighborhoods $\delta, 2\delta, ..., k\delta$), as well as the number of neighboring points of x $(h, h+qh, ..., h+(k-1)qh)$, such that $h+(t-1)qh$ neighboring points $(t \in \{1, ..., k\})$ are analyzed in the $t\delta$-neighborhood of x.

The degree of robustness contributes for the evaluation of a solution (individual) of a population and enables to classify the solutions according to their level of robustness with respect to changes in the decision variable values.

2.2 Perturbations in the objective function coefficients

For this case, it is assumed that perturbations of the objective function coefficients may occur in any coefficient of objective function f_r $(c_{r1}, c_{r2}, ..., c_{rn})$, for $r = 1, ..., R$.

The concept of scenario is introduced, which is a possible representation of the objective function coefficients. The set of initial (nominal) objective function coefficients is denoted by reference scenario. The assessment of the degree of robustness of a solution x involves analyzing the neighborhood of the reference scenario s, where x is a solution to the problem and $f^s(x)$ is the image of x in the objective space for the reference scenario s.

The underlying idea is to determine a set of neighborhoods $k\delta$ around the reference scenario s, such that the images of x for these neighborhood scenarios are better than $f^s(x)$ or belong to a pre-specified neighborhood η around $f^s(x)$ in the objective space. The process begins by analyzing randomly generated scenarios inside a hyperbox of radius δ around s. This neighborhood (hyperbox) is then progressively enlarged, in multiples of δ (δ, 2δ, ...), until the percentage of scenarios for which the images of x in the objective space are better than $f^s(x)$ or belong to the neighborhood of $f^s(x)$ is lower than a pre-defined threshold. This enables to assign a degree of robustness to solutions according to the number of hyperbox enlargements for which that condition is fulfilled (see Fig. 2).

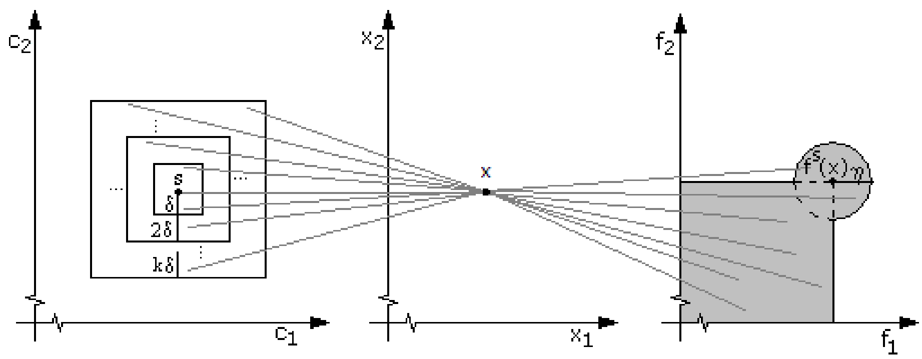

Figure 2: Definition of neighborhoods in the scenario space and objective function space associated with a solution x (for 2-dimension spaces and all functions to be minimized).

The degree of robustness depends on the size of a δ-neighborhood of scenario s and the percentage of the h neighboring points whose objective function values for x are better than $f^s(x)$ or belong to the η-neighborhood of $f^s(x)$. Those h neighboring points are randomly generated around scenario s (see also Barrico and Antunes [3, 4], and Deb and Gupta [11]).

The degree of robustness of solution x is a value k, such that (see Fig. 2):

1) the percentage of scenarios s' in the $k\delta$-neighborhood of s, for which the objective function values $f^{s'}(x)$ that are better than $f^s(x)$ or belong to the η-neighborhood of $f^s(x)$, is greater than or equal to a pre-specified threshold p;

2) the percentage of scenarios s' in the $(k+1)\delta$-neighborhood of s, for which the objective function values $f^{s'}(x)$ that are better than $f^s(x)$ or belong to the η-neighborhood of $f^s(x)$, is lower than p.

The degree of robustness k of a solution x determined in the reference scenario s is gradually computed as k increases (neighborhoods δ, 2δ, ..., $k\delta$), as well as the number of neighboring points of s $(h, h+qh, ..., h+(k-1)qh)$, such that $h+(t-1)qh$ neighboring points ($t \in \{1, ..., k\}$) are analyzed in the $t\delta$-neighborhood of s.

2.3 Robustness parameters

The h parameter is associated with the number of points to be analyzed in the neighborhood of a solution or reference scenario.

The δ parameter is associated with the base amplitude of the neighborhood of a solution or reference scenario. The neighborhoods are then enlarged in multiples of δ: δ, ..., $k\delta$.

The parameter p may be perceived as an indicator of the robustness exigency.

In problems where the perturbations are in the decision variable space, if $p = 100\%$ then the objective function values of all neighboring solutions of x are better than $f(x)$ or belong to the predefined η-neighborhood of $f(x)$. If $p = 90\%$ then the number of neighboring solutions of x, for which the objective function values are better than $f(x)$ or belong to the η-neighborhood of $f(x)$ is at least 90%.

In the problems where the perturbations are in the objective function coefficients, if $p = 100\%$ then all $f^s(x)$ in all neighboring scenarios s' tested are better than $f^{s'}(x)$ or belong to the predefined η-neighborhood of $f^s(x)$. If $p = 90\%$ then the $f^{s'}(x)$ that are better than $f^s(x)$ or belong to the η-neighborhood of $f^s(x)$ are at least 90%.

So, it is more probable that a solution x with degree of robustness k in the first case ($p = 100\%$) has actually this degree of robustness than in the second case ($p = 90\%$) when p is relaxed.

The parameter η is used as a upper bound of the distance between two solutions (of any type) in the objective space (for perturbations in the decision variable space) or between the images of a solution according to two scenarios (for perturbations in the objective function coefficients). This parameter reflects the indifference thresholds regarding the values of each objective function. As the value of η increases, meaning that the DM is more tolerant with respect to differences in objective function values, the trend is that the number of more robust solutions also increases. The location of the more robust solutions on the non-dominated front is a relevant insight to aid the DM in the selection of a compromise solution.

The q parameter value is a coefficient ($0 \leq q \leq 1$) associated with the increase of the number of neighboring points to be analyzed in the successive enlargements of the neighborhood of a solution or reference scenario (k^{th} neighborhood, with $k > 1$). For example, if $q = 0.5$ then the number of points to be analyzed in the k^{th} neighborhood is increase by $h/2$ relatively to the previous neighborhood: δ (h points), 2δ ($h + h/2$ points), 3δ ($h + h/2 + h/2$ points), ...

However, the p, η and q parameter values may be different depending on the solution type (if solutions are non-dominated, dominated or infeasible). Due to the considerable run time of the evolutionary algorithm, we recommend that the p parameter value should be higher for the dominated and infeasible solutions, and η and q parameters values should be higher for the non-dominated solutions. The underlying rationale is that non-dominated solutions are more important than dominated and infeasible solutions, and therefore a more exhaustive analysis is necessary for the non-dominated solutions.

2.4 Metrics

To analyze whether, for a solution y belonging to δ-neighborhood of solution x, $f(y)$ belongs to the η-neighborhood of $f(x)$, it is necessary to calculate the distance between the images of solutions x and y in the objective space, $f(x)$ and $f(y)$. In the same way, to analyze whether, for a scenario s' belonging to the δ-neighborhood of scenario s, $f^{s'}(x)$ belongs to the η-neighborhood of $f^s(x)$, it is necessary to calculate the distance between the images of solution x according to the scenarios s' and s, $f^{s'}(x)$ and $f^s(x)$, in the objective space.

The normalized distance between $f(x)$ and $f(y)$ is computed by the expressions $||f(y) - f(x)|| \ / \ ||f(x)||$ (relative) or $||f(y) - f(x)||$ (absolute), and the normalized distance between $f^{s'}(x)$ and $f^s(x)$ is computed by the expression $||f^{s'}(x) - f^s(x)||/||f^s(x)||$ (relative) or $||f^{s'}(x) - f^s(x)||$ (absolute), where the operator $||.||$ can be any suitable metric, such as the city block, Euclidean or Chebycheff metrics. The choice of the metric can also have a role to play: in the city block metric all differences have the same importance, whereas in the Chebycheff metric only the greatest difference in all dimensions matters. Therefore, the Chebycheff metric capture better a situation in which non-compensation is an important issue.

The non-normalized distance between $f(x)$ and $f(y)$, or between $f^{s'}(x)$ and $f^s(x)$, can be used too. In these cases, $\eta = (\eta_1, ..., \eta_R)$, where R is the number of objective functions of the problem (see Fig. 2, for $R = 2$).

3 The evolutionary algorithm

An evolutionary algorithm encompassing the assessment of the degree of robustness associated with each solution has been implemented, which uses an elitist strategy with a secondary population (with feasible non-dominated solutions only) of constant maximum size. The elitist strategy is aimed at increasing the performance, both accelerating the convergence speed towards the non-dominated frontier and ensuring the solutions attained are indeed non-dominated ones and well spread over the frontier. This is an important issue in real-world problems (see Gomes et al. [15]) since it is necessary to provide the DM with well-distributed and diverse solutions for a well-informed final decision to be made upon.

In order to bias the evolutionary process towards more robust solutions, this concept of degree of robustness is imbedded into the fitness assessment of each individual jointly with the non-dominance test. In each non-dominance level, more robust solutions are more likely to contribute for the next generation.

The main steps of this algorithm are the following:

- The fitness of the individuals composing the main population is computed;
- From the main population (consisting of POP individuals) $POP - E$ individuals are selected by using a tournament technique (E is the size of the elite set);
- A new population is formed by the $POP - E$ offspring generated by crossover and mutation, and E individuals (elite) that are the more robust in the secondary population;
- The evaluation of individuals is carried out by using a dominance test and their degree of robustness, defining an approximation to the non-dominated frontier;
- The non-dominated solutions are computed and they are processed to update the secondary population using a sharing technique, if necessary.

3.1 Fitness assessment

The fitness value of a solution depends on its degree of robustness and the dominance test. For each solution, the fitness computation uses a "non-dominated sorting" technique as in "NSGA-II" (Deb [10] and Deb et al. [13]), and involves determining various solution fronts in the following way:

- The first front consists of all non-dominated solutions, a minimum fitness value equal to $POP \times (MaxDegree + 1)$ being assigned to them;
- This fitness value of each one of these solutions is incremented by its degree of robustness;
- The solutions in the first front are temporarily ignored, and the remaining feasible solutions (the dominated solutions) are processed by applying them a dominance test (the non-dominated solutions will belong to the second front);

- The minimum fitness value of the current front is obtained by subtracting $MaxDegree+1$ to the minimum fitness value of the previous front, which is assigned to the solutions of the current front;

- For each solution in the current front, the fitness value is incremented by its degree of robustness;

- This process continues until all feasible solutions are assigned a fitness value;

- The same process is repeated for the infeasible solutions until all infeasible solutions are assigned a fitness value.

If two solutions have the same fitness value, then the best solution is the one with fewer solutions in its neighborhood, according to a defined radius. A niche is defined by a radius *df* around a solution, where *df* is the maximum distance between solutions necessary to obtain a well-spread front. For example, for bi-objective problems the *df* value is equal to $MaxDist/POP$, where $MaxDist$ is the distance between the pseudo-solutions obtained by considering the best and the worst values for each objective function in the main population.

This procedure is aimed at ensuring that the solutions of the k dominance level have a fitness value greater than the solutions of the $k + 1$ level, and in the same level the solutions with the higher degree of robustness have a higher fitness value. Finally, in the group of solutions with the same dominance level and equal degree of robustness, the best solutions in the group are the ones with fewer neighbors in the population.

3.2 The sharing mechanism

The sharing mechanism for updating the secondary population uses a niche scheme with a radius of dynamic value. This mechanism is applied after computing all non-dominated solutions candidate for the secondary population. These are all the non-dominated solutions in the set formed by the secondary population and the main population. This mechanism is only applied when the number of solutions candidate for the secondary population ($NCPS$) is greater than the size of this population (NPS).

The sharing mechanism consists in the following steps (adapted for bi-objective problems):

1) Insert the extreme solutions (those with the best values for each objective function);

2) Compute the first niche radius (ds) as the ratio: normalized distance between extreme solutions / NPS (that is, $\sqrt{2}/NPS$);

3) Insert solutions located at a distance greater than ds from the ones already belonging to the secondary population, using the degree of robustness as the priority criterion;

4) Update the value of the niche radius, ds, by reducing it by 10%;

3.3 The initial population

The strategy used to determine the initial population consists in randomly generating feasible non-dominated solutions only. This process consists in the following steps:

1) Randomly generate a feasible solution;
2) If this solution is non-dominated with respect to the initial population then insert this solution into the initial population and apply the dominance test to update this population;
3) If the initial population is not complete then return to step 1.

However, in problems with few non-dominated feasible solutions the initial population may also contain feasible dominated solutions. For this purpose a counter of generated solutions is used. When this counter attains a pre-specified number of generated solutions and the initial population is not complete, then the next feasible solutions generated are inserted into the initial population until it is complete.

3.4 The algorithm

The evolutionary algorithm consists in the following steps:

1) Initialization: randomly generate the initial population with POP non-dominated solutions (see also 3.3);
2) Determine the degree of robustness of each individual in the initial population;
3) Evaluation: compute the fitness value of each individual in the initial population, using its degree of robustness;
4) Determine the (initial) secondary population of size NPS from the initial population: if $NPS \geq POP$ then copy all solutions from the initial population to the secondary population; else apply the sharing mechanism to the initial population to select NPS solutions;
5) Current population $\leftarrow$ initial population;

   Repeat

6) Build up the (main) population associated with the next generation of size POP:
   a) Introduce E more robust individuals from the secondary population (elite) into the main population;
   b) Select 2 individuals of the current population by tournament (in each tournament, 10% of the individuals in this population are used to generate the selected one);
   c) Apply genetic crossover and mutation operators to the 2 individuals selected;

d) Insert the new individuals into the main population;

e) If the main population does not yet contain POP individuals then return to step b);

7) Determine the degree of robustness of each individual in the main population;

8) Evaluation: apply the dominance test and compute the fitness value of each individual in the main population, using its degree of robustness;

9) Determine the $NCPS$ solutions candidate to becoming part of the secondary population;

10) Update the secondary population: if $NPS \geq NCPS$ then copy all candidate non-dominated solutions to the secondary population; else apply the sharing mechanism to all solutions found in step 9 to select NPS solutions;

11) Current population $\leftarrow$ main population;

Until the pre-specified number of iterations is attained.

4 Perturbations in the decision variable space - illustrative examples

In this section examples are presented to illustrate the application of the concept of degree of robustness in the framework of the evolutionary algorithm described in the section 3. The main goal of this evolutionary algorithm is to characterize the non-dominated front containing the more robust non-dominated solutions and also ensuring its diversity along the front.

Each solution x is assigned a degree of robustness, which depends on the behavior of $f(x)$ for solutions within a neighborhood of x.

Users can specify the size of the solution neighborhood, both in the decision variable space and in the objective function space.

The aim is to study the influence of parameters p and η in the determination of the Pareto front of robust non-dominated solutions. These parameters are associated with the exigency level of robustness and the indifference threshold between the objective function values specified by the DM/analyst.

The population used in the algorithm implementation associated with the two test problems analyzed in this section consists of individuals represented by an array of $M = M_1 + ... + M_n$ binary values, where n is the number of decision variables and M_i is the binary size of the i^{th} decision variable representation.

4.1 Test problem 1

This test problem is commonly used in the engineering design optimization literature (see also Deb [10] and Li et al. [19]). This problem is formulated as follows:

Minimize $f_1(x) = x_1\sqrt{16 + x_3^2} + x_2\sqrt{1 + x_3^2}$

Minimize $f_2(x) = 20\sqrt{16 + x_3^2}/(x_1 x_3)$

Subject to

$$20\sqrt{16 + x_3^2} - 100x_3x_1 \leq 0,$$
$$80\sqrt{1 + x_3^2} - 100x_3x_2 \leq 0,$$
$$x_1 > 0,$$
$$x_2 \geq 0,$$
$$1 \leq x_3 \leq 3$$

In the binary representation of the decision variables $M_1 = M_2 = 17$ and $M_3 = 6$, and $\Delta x = (0.0001,\ 0.0001,\ 0.05)$ where Δx is the variation in the design decision variables. Thus, by construction M_i zeros correspond to the lower bound of x_1 (Δx_1), x_2 (0) and x_3 (1), and M_i ones correspond to the maximum value for x_1 (13.1071), x_2 (13.1071) and x_3 (4.15) in the evolutionary algorithm. These are the satisfactory maximum values for an accurate analysis.

Uniform crossover and binary mutation have been used, with probability pc and pm, respectively.

The evolutionary algorithm for determining the non-dominated front uses the following parameter values: $POP = 120$; $NPS = 60$; $E = 0.1NPS$; $pc = 0.95$; $pm = 0.1$; and *number of iterations* $= 20$.

4.1.1 Non-dominated front

Fig. 3 displays the non-dominated front obtained without considering robustness analysis.

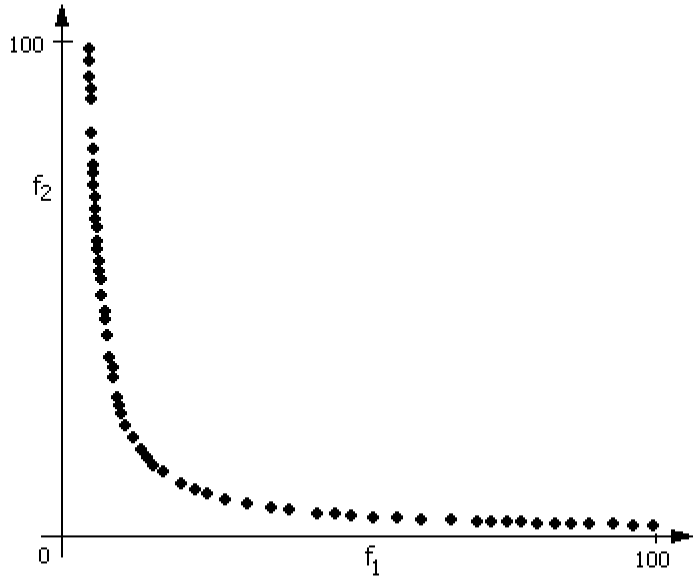

Figure 3: Pareto-front (without robustness analysis).

Fig. 4 displays the non-dominated front obtained with the following parameter values associated with the robustness analysis: $p = 100\%$; $h = 100$; $\delta = (\delta_1, \delta_2, \delta_3) = (0.1, 0.02, 0.05)$; $\eta = 0.75$, which is used as a upper bound of the absolute normalized distance between two solutions (of any type) in the objective space; and $q = 1$, for all solution types.

In this front (Fig. 4), non-dominated solutions are categorized into 2 degrees of robustness (0 and 1). This information is relevant from a decision support point of view, since a DM would prefer a non-dominated compromise solution displaying a higher degree of robustness. In this case, with the maximum level of exigency ($p = 100\%$) the most robust solutions are located in the "central" region of the non-dominated front (well-balanced within the range of non-dominated solutions), slightly extending along the non-dominated frontier towards a slight improvement of f_2 at an expense of degrading the value of f_1.

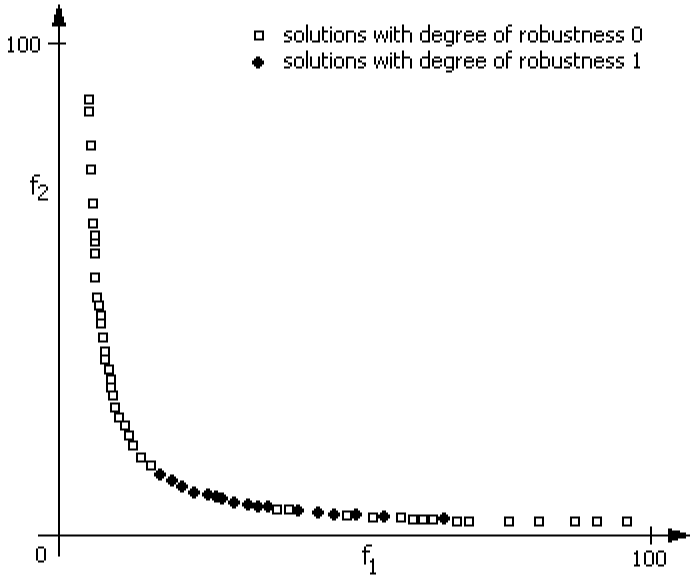

Figure 4: Pareto-front (with robustness analysis, $p = 100\%$, and $\eta = 0.75$).

4.1.2 Analysis of parameter p

In this section the robustness parameter p will be the object of the analysis. All the other robustness parameter values are the same used to obtain the front displayed in Fig. 4 (in which $p = 100\%$).

Fig. 5 - Fig. 7 display the non-dominated solutions obtained with different p values for non-dominated solutions: 90% (Fig. 5), 70% (Fig. 6) and 50% (Fig. 7).

In these examples, non-dominated solutions are categorized into 2 degrees of robustness with $p = 90\%$ (such as with $p = 100\%$), 3 different degrees of robustness with $p = 70\%$, and 4 different degrees of robustness with $p = 50\%$ (Fig. 5 - Fig. 7). For dominated feasible and infeasible solutions $p = 100\%$.

As p decreases, from 100% to 90%, some solutions with a degree of robustness 0 become of degree of robustness 1, reinforcing the trend of extending along the non-dominated frontier towards a slight improvement of f_2 (in a region where f_2 values are not far from its optimum) in exchange of degrading the value of f_1 (Fig. 5).

With $p = 70\%$ (Fig. 6), non-dominated solutions with the higher degree of robustness are located closer to the optimum of f_2. In this case there are non-dominated solutions possessing a degree of robustness 2, which did not appear in the transition of p from 100% to 90%.

Finally, in the transition of p from 70% to 50% (Fig. 7), there is an increase of the number of solutions with degree of robustness 2 also closer to the optimum of f_2. In this front some solutions with degree of robustness 3 appear, which did not happen in the previous examples with higher values for the p parameter. These solutions are broadly located in the same regions where the solutions with degree of robustness 2 are located in the front obtained in the previous example (with $p = 70\%$), that is closer to the optimum of f_2.

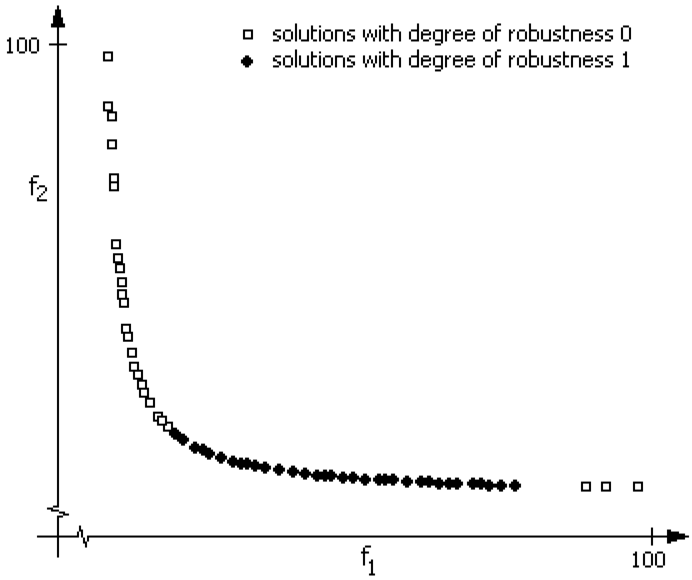

Figure 5: Pareto-front ($p = 90\%$).

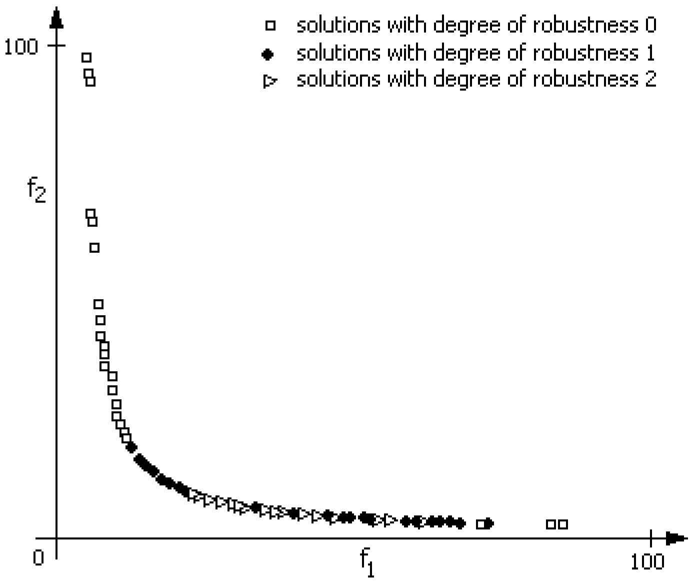

Figure 6: Pareto-front ($p = 70\%$).

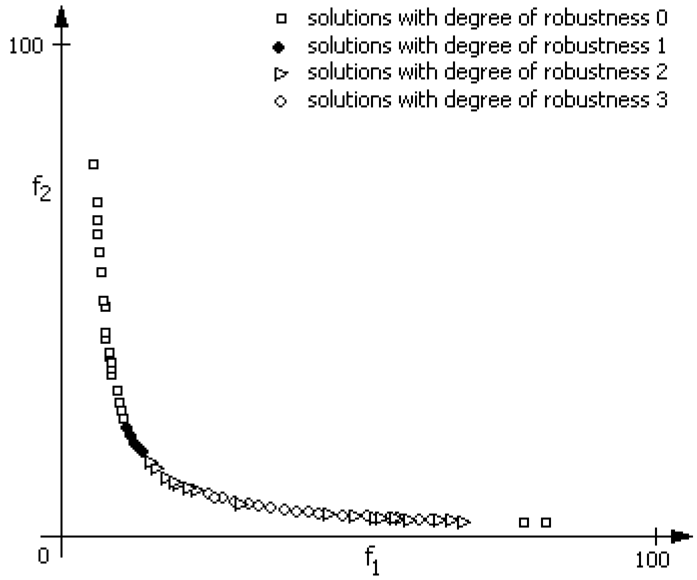

Figure 7: Pareto-front ($p = 50\%$).

This study provides the DM thorough information regarding the selection of a compromise solution (between the objective function values) which also presents a high level of robustness. Non-dominated solutions which privilege f_1 possess a low degree of robustness. A higher degree of robustness is achieved in a region of well-balanced solutions, but closer to the optimum of f_2. This information is relevant for a DM when assessing the merits of competing non-dominated solutions, not just examining the trade-off between the objective function values but also their degree of robustness.

4.1.3 Analysis of parameter η

In this section the influence of parameter η on the degree of robustness assigned to solutions is analyzed. η contains the upper limits of the absolute normalized distance between two solutions (of any type) in the objective space. All the other robustness parameter values are the same used to obtain the front displayed in Fig. 4 (in which $\eta = 0.75$).

Fig. 8 - Fig. 10 display the non-dominated fronts obtained with different η values: 1.0 (Fig. 8), 1.5 (Fig. 9) and 0.5 (Fig. 10).

In the transition of η from 0.75 to 1.0 (Fig. 8), the non-dominated front obtained is composed by solutions with 2 degrees of robustness (0 and 1), such as in the front obtained with $\eta = 0.75$ (Fig. 4). However, there is an increase of the number of solutions with a higher degree of robustness (1). This happens because the increase of the η value implies relaxing the importance assigned to differences between the objective function values (up to a threshold) and thus enlarging the neighborhood in which the DM is indifferent between non-dominated solutions located therein.

In the transition of η from 1.0 to 1.5 (Fig. 9), some solutions of degree of robustness 1 become of degree of robustness 2.

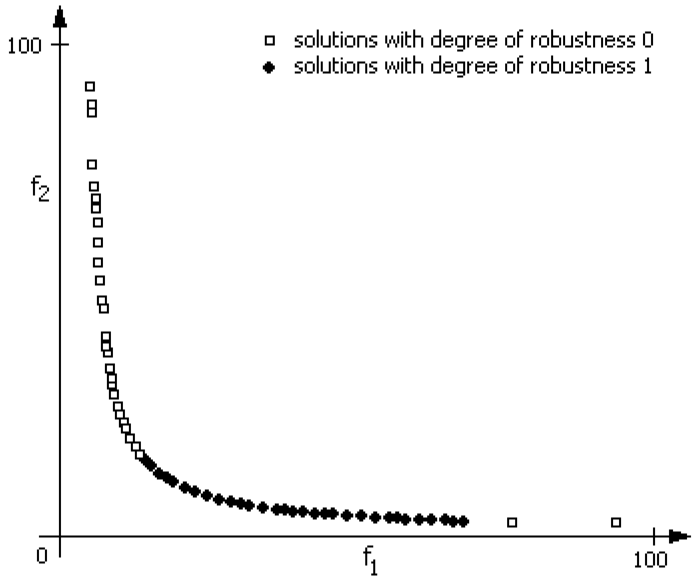

Figure 8: Pareto-front ($\eta = 1.0$).

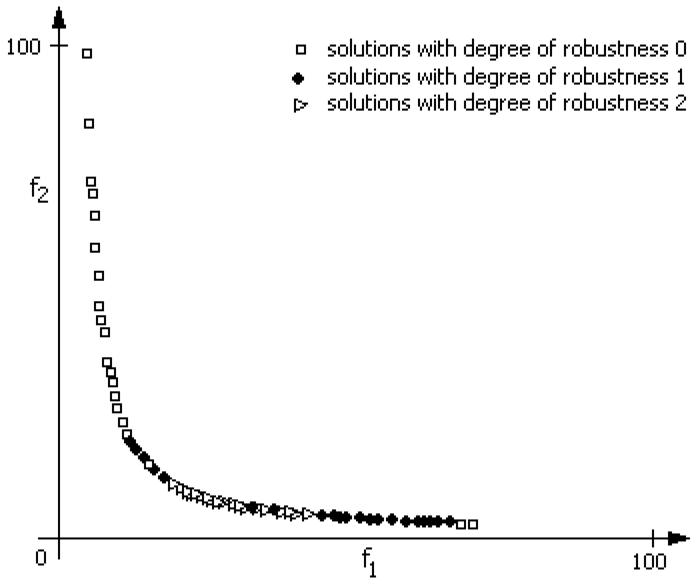

Figure 9: Pareto-front ($\eta = 1.5$).

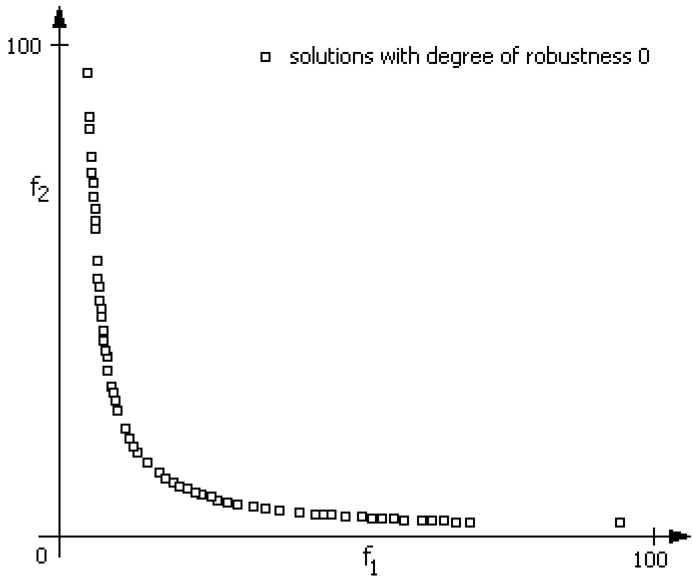

Figure 10: Pareto-front ($\eta = 0.5$).

In the opposite sense, with the decrease of η from 0.75 to 0.5 (Fig. 10), all non-dominated solutions of the front obtained have the same degree of robustness (0). In this case, differences in the objective function values become more important for the DM (that is, his/her indifference threshold for considering irrelevant for discriminating purposes the difference between the objective functions values decreases). This means that the amplitude of the neighboring solutions in the objective space, with respect to the solution under analysis, also decreases.

Therefore, solutions with a higher degree of robustness are mostly located in a region of the non-dominated front presenting values for f_2 not far from its optimal value and values for f_1 in the mid of the range of the values f_1 attains within the non-dominated front. This type of insights provided by the assignment of a degree of robustness to the non-dominated solutions is a relevant information for a DM to assess their merits in the selection of a satisfactory compromise solution.

4.1.4 Using other metrics

Also, other type of metrics can be used in the robustness analysis, such as a relative normalized and absolute non-normalized distance between two solutions in the objective space.

In Fig. 11, a relative normalized distance has been used with the same parameter values used to obtain the front displayed in the Fig. 4, except the η parameter value ($\eta = 0.05$, this mean 5% of the reference value). In this case, the most robust solutions are still located towards the best values for f_2, but without sacrificing f_1 too much.

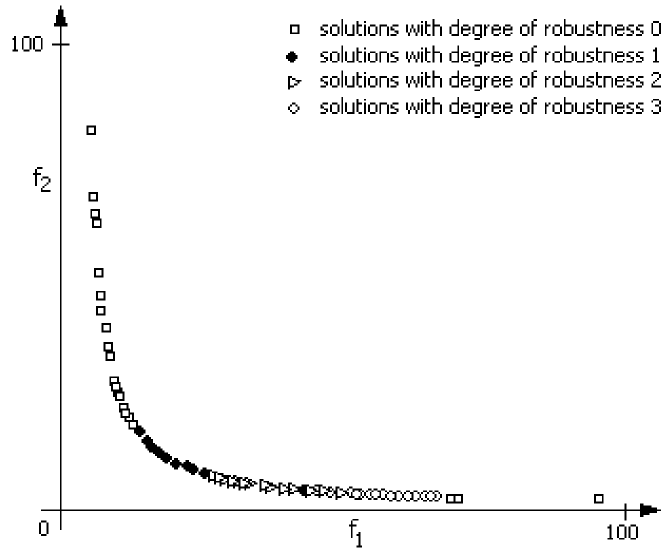

Figure 11: Pareto-front obtained by using the relative normalized distance.

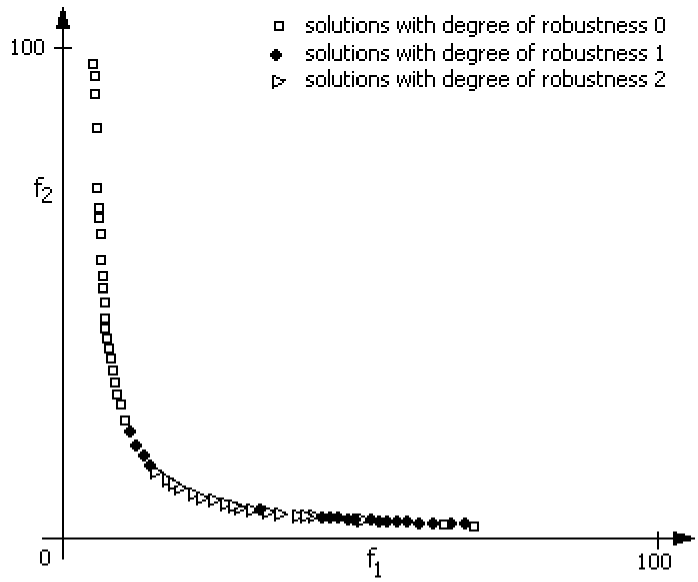

Figure 12: Pareto-front obtained by using the absolute non-normalized distance.

In Fig. 12, an absolute non-normalized distance has been used with the same parameter values used to obtain the front displayed in Fig. 4, except the η value: $\eta = (\eta_1, \eta_2) = (1.5, 1.5)$. The most robust solutions possess the same characteristics mentioned before: they present very good values for f_2, and values for f_1 in the mid of its range within the non-dominated front.

The specification of parameter η does not impose an excessive burden on the DM since he/she is familiar with the objective function space. Therefore, he/she is able to provide information on the thresholds below which the difference between solutions is not relevant for decision purposes.

4.2 Test problem 2

The second test problem has also been studied in Deb and Gupta [11, 12]. This problem is formulated as follows:

Minimize $f_1(x) = x_1$

Minimize $f_2(x) = h(x_1) + g(x)S(x_1)$

Subject to

$0 \leq x_1 \leq 1$

$-1 \leq x_i \leq 1$, i = 2, ..., 5

Where

$h(x_1) = 1 - x_1^2$

$g(x) = \sum_{i=1}^{n} 10 + x_i^2 - 10cos(4\pi x_i)$

$S(x_1) = \frac{1}{0.2+x_1} + x_1^2$

In the binary representation of the decision variables $M_i = 30$ (with $i = 1, ..., 5$) and all generated solutions are feasible solutions because, by construction, M_i zeros correspond to the lower bound and M_i ones correspond to the upper bound of the interval variation of the i^th decision variable, and these are the only constraints of the problem.

Uniform crossover and binary mutation have been used, with probability pc and pm, respectively.

The evolutionary algorithm for determining the non-dominated front uses the following parameter values: $POP = 100$; $NPS = 80$; $E = 0.1NPS$; $pc = 0.9$; $pm = 0.35$; and *number of iterations* = 10000.

For this test problem, only the robustness parameter p is analyzed.

4.2.1 Non-dominated front

Fig. 13 displays the non-dominated front obtained without considering robustness analysis.

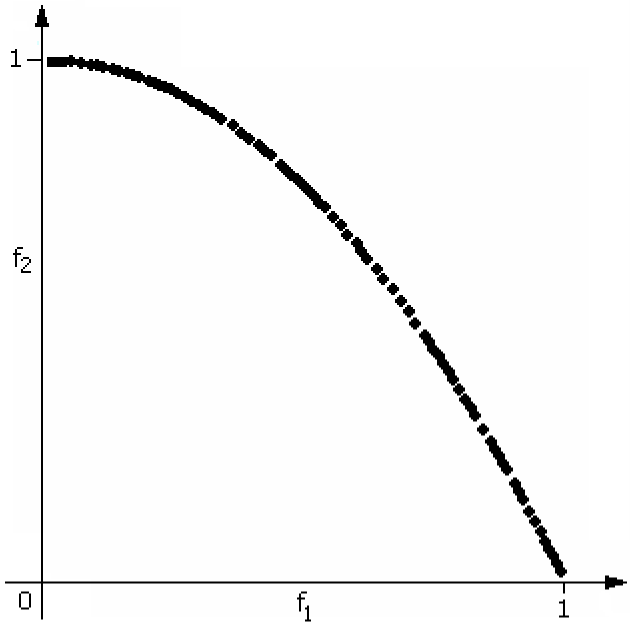

Figure 13: Pareto-front (without robustness analysis).

4.2.2 Analysis of parameter p

Fig. 14 - Fig. 16 show the non-dominated solutions obtained with different pre-defined p values for the non-dominated solutions: 100% (Fig. 14), 85% (Fig. 15) and 60% (Fig. 16). The values for the other robustness parameters are: $h = 100$; $\delta = (\delta_1, ..., \delta_)$ $= (0.004, 0.008, 0.008, 0.008, 0.008)$; $\eta = 0.7/0.1$ (non-dominated / dominated solutions), which is used as the upper limit of the relative normalized distance between two solutions in the objective space; $q = 0.1/0$ (non-dominated / dominated solutions); and $p = 100\%$ (dominated solutions).

Non-dominated solutions are just categorized into 2 degrees of robustness, even decreasing the level of exigency represented by p from 100% to 60% (Fig. 14 - Fig. 16): 0 and 1 ($p = 100\%$); 1 and 2 ($p = 85\%$ and $p = 60\%$). With $p = 100\%$ robust solutions are found throughout the entire non-dominated frontier, except near the optimum of f_1.

As p decreases, thus decreasing the level of exigency of the robustness, some non-dominated solutions are now classified as having a degree of robustness 2, and some solutions that had a degree of robustness 0 possess now a degree of robustness 1. Also, more well-balanced solutions, regarding the values of f_1 and f_2, present a higher degree of robustness as the exigency decreases. These solutions are likely to present acceptable trade-offs for a DM in order to be selected as satisfactory compromise solutions.

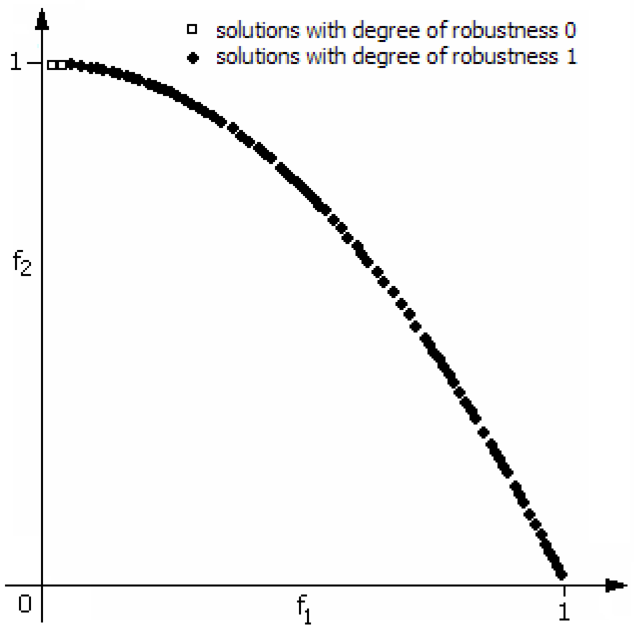

Figure 14: Pareto-front ($p = 100\%$).

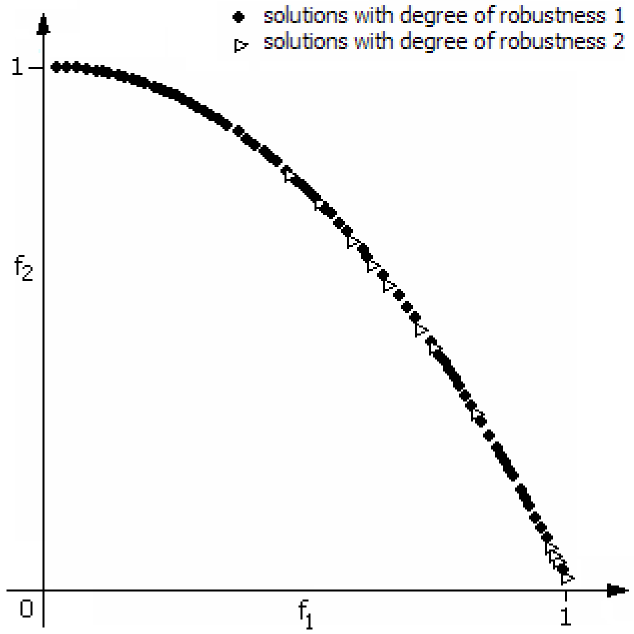

Figure 15: Pareto-front ($p = 85\%$).

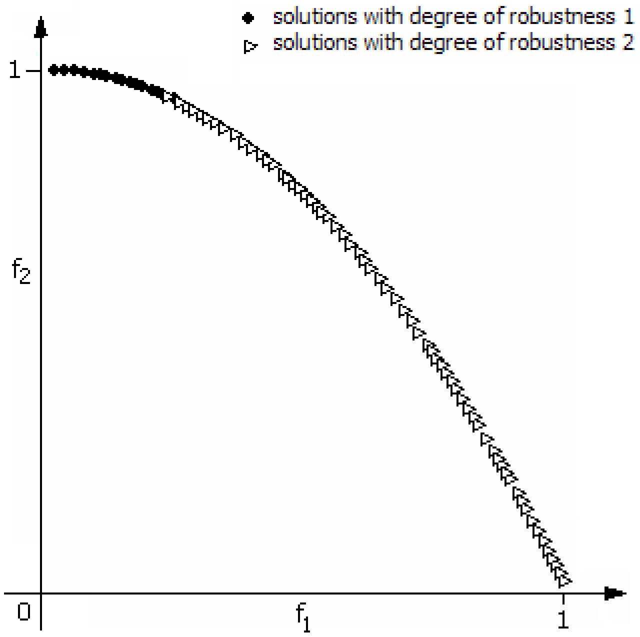

Figure 16: Pareto-front ($p = 60\%$).

In general, it is expected that a DM strives for well-balanced solutions (that is, presenting trade-offs with satisfactory values for both objective functions) and displaying a high degree of robustness. The information on the degree of robustness can be used to complement the evaluation of the merit of non-dominated solutions based on the objective function values.

5 Perturbations in the objective function coefficients - an example of a reactive power compensation problem

In this section, the approach to robustness analysis in multi-objective evolutionary optimization for tackling perturbations in the objective function coefficients is illustrated. This approach relies on the concept of scenario and reference scenario (see section 2.2). It is based on the solution behavior in the neighborhood of the reference scenario, when the objective function coefficients change from their nominal values.

The concept of degree of robustness permits the user to exert a control on the desired level of robustness of solutions obtained, such as it was illustrated in section 4. Users can specify the size of the reference scenario neighborhood, both regarding the objective function coefficients and the objective function values.

The evolutionary approach described in section 3 has been adapted to the reactive power compensation problem in electrical distribution networks (see Antunes at al. [1]), incorporating the concept of degree of robustness.

The mathematical model for the reactive power compensation problem in electrical distribution networks considering two objective functions is presented. The evolutionary and illustrative results are then described.

5.1 The reactive power compensation problem in electrical distribution networks

Reactive power compensation is an important issue in electric power systems, involving operational, economical and quality of service aspects. Consumer loads (residential, industrial, service sector, etc.) impose active and reactive power demand, depending on their characteristics. Active power is converted into "useful" energy, such as light or heat. Reactive power must be compensated to guarantee an efficient delivery of active power to loads, thus releasing system capacity, reducing system losses, and improving system power factor and bus voltage profile. The achievement of these aims depends on the sizing and location of shunt capacitors (sources of reactive power).

The problem of reactive power compensation involves determining the number, location, and sizes for shunt capacitors (sources of reactive power) to be installed, in this case in a distribution feeder, to achieve a balance between costs (associated with installing new capacitors), and technical and quality of service evaluation aspects.

The model used in this study explicitly assumes the multi-objective nature of the problem by considering two (conflicting) objective functions: minimizing (resistive) losses and minimizing the installation costs of new sources of reactive power. Constraints are related with requirements of acceptable node voltage profile (quality of service that result from legislation), power flow (physical laws in electrical networks), and impossibility of capacitor locations at certain nodes (technical restrictions). For more details about this mathematical model see Antunes et al. [1] and Pires et al. [23].

The input data to feed the model are the following: installation costs of each capacitor type, resistance and inductance of each part of the distribution network, active and reactive power (load) at each node of the distribution network, and the value of the capacity of each capacitor to be installed.

In this study, it is assumed that the data associated to costs and capacities of capacitors are unchanged and the remaining data is subject to small perturbations, due to the uncertainties inherent to measurements and estimates.

5.2 The evolutionary algorithm

This evolutionary approach, incorporating robustness analysis to characterize the non-dominated front for a reactive power compensation problem in electrical distribution networks, has been applied to a radial distribution system with 28 nodes and 6 lateral buses (see Das et al. [9]). Three types of capacitors are considered for possible installation.

The population used in the algorithm implementation consists of individuals represented by an array of NN integer values (NN being the number of network nodes where it is possible to install a new capacitor or change the capacity of a capacitor already installed). The index of the array corresponds to a network node and the value therein denotes the type of capacitor to install in that node (the capacitor type is indexed by an index ranging from 0 to J - where 0 means no capacitor; that is, J different capacitor sizes can be installed).

Uniform crossover has been used, with probability *pc*. In each generation of the algorithm a new mask is created. The capacitor type is indexed by an index ranging from 0 to J. The mutation consists in modifying (with a probability *pm*) the current index value to one of other possible values.

This evolutionary algorithm uses the following parameter values (Fig. 17 - Fig. 24): $POP = 40$; $NPS = 30$; $E = 0.1NPS$; $pc = 0.95$; $pm = 0.05$; and *number of iterations* = 2000.

The goal is to study the influence of the parameters p and η in the determination of the robust non-dominated solution front. These parameters are associated with the exigency level of robustness specified by the DM/analyst. The parameter η is used as upper bound of the absolute non-normalized distance between the images in the objective space of the solutions (of any type) associated with two distinct scenarios.

5.3 Non-dominated front

The Fig. 17 shows the Pareto-front obtained without considering the degree of robustness.

Fig. 18 displays the non-dominated front obtained with the following robustness parameter values: $p = 100\%$, for all solution types; $h = 100$; $\delta = 0.003$ for resistance and inductance of branches and $\delta = 0.03$ for active and reactive power at nodes; $\eta = (\eta_1, \eta_2) = (0.004, 150)$; $q = 1$, for all solution types.

For $p = 100\%$ most solutions have degree of robustness 1 and a few solutions near the optimum of the cost of objective function have a degree of robustness 0.

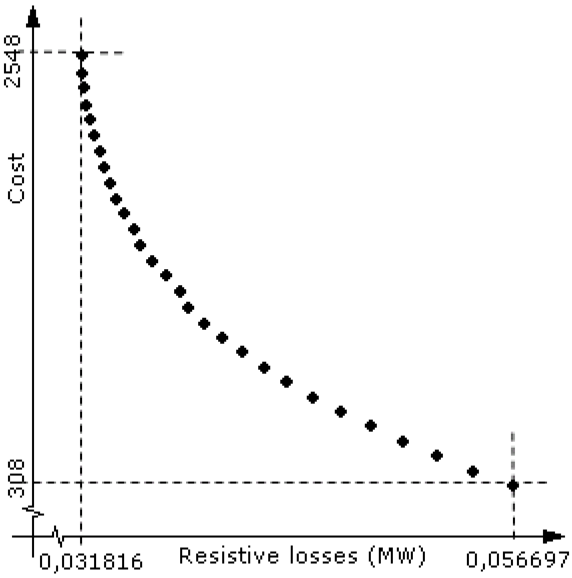

Figure 17: Pareto-front obtained without considering robustness analysis.

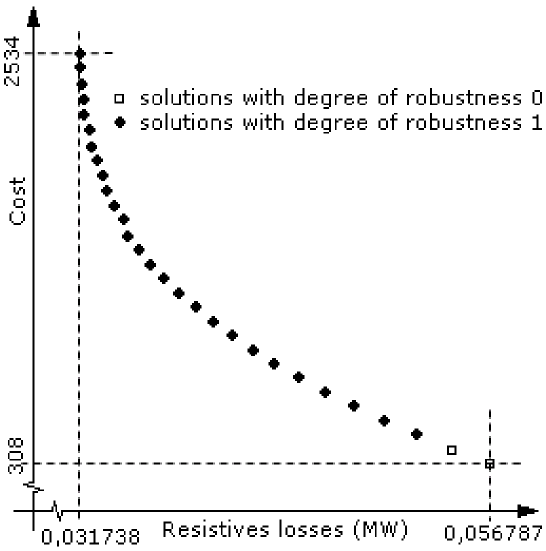

Figure 18: Pareto-front with robustness analysis: $p = 100\%$, and $\eta = (\eta_1, \eta_2) = (0.004, 150)$.

5.4 Analysis of parameter p

Fig. 19 - Fig. 21 display the non-dominated solutions obtained with different pre-defined p values for the non-dominated solutions: 95% (Fig. 19), 90% (Fig. 20), and 80% (Fig. 21). For the dominated feasible and infeasible solutions $p = 100\%$. The values of other parameters (associated with the robustness analysis) are the same used to obtain the front displayed in Fig. 18, where $p = 100\%$.

As p decreases, thus decreasing the level of exigency of the robustness, the non-dominated solutions present a higher degree of robustness. The most robust solutions are generally located towards the best values for the resistive losses objective function. The relaxation of p generally enables to obtain a better discrimination regarding robustness for the solutions in the Pareto-front.

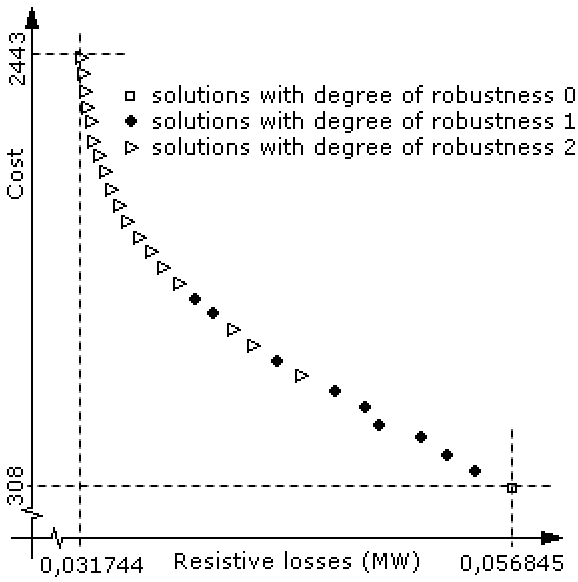

Figure 19: Pareto-front obtained with $p = 95\%$.

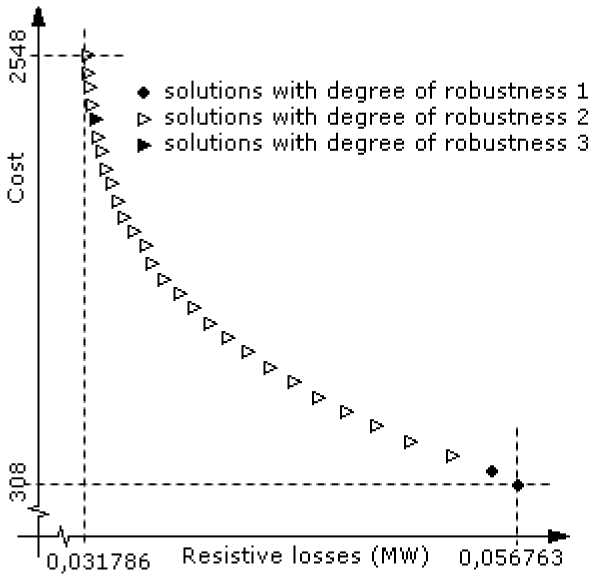

Figure 20: Pareto-front obtained with $p = 90\%$.

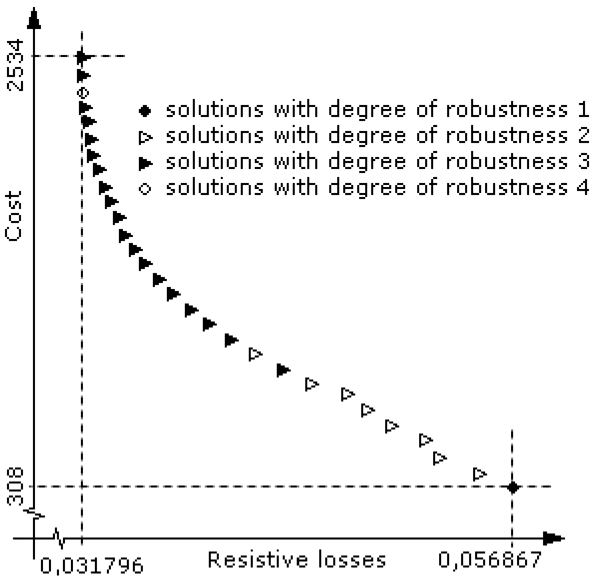

Figure 21: Pareto-front obtained with $p = 80\%$.

5.5 Analysis of parameter η

Fig. 22 - Fig. 24 display the non-dominated solutions obtained with different pre-defined η values for any solution type (non-dominated, dominated and infeasible solutions): $\eta_1 = 0.003$ and $\eta_2 = 100$ (Fig. 22), $\eta_1 = 0.005$ and $\eta_2 = 200$ (Fig. 23), and $\eta_1 = 0.006$ and $\eta_2 = 250$ (Fig. 24). The values of other parameters (associated with the robustness analysis) are the same used to obtain the front displayed in Fig. 18, where $\eta = (\eta_1, \eta_2) = (0.004, 150)$.

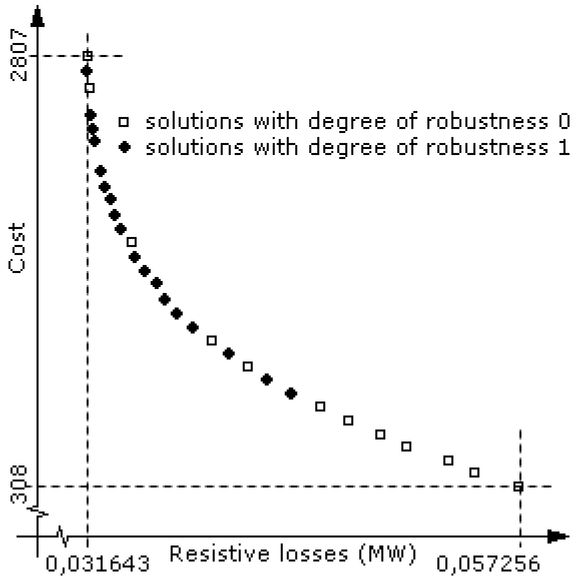

Figure 22: Pareto-front obtained with $\eta = (0.003, 100)$.

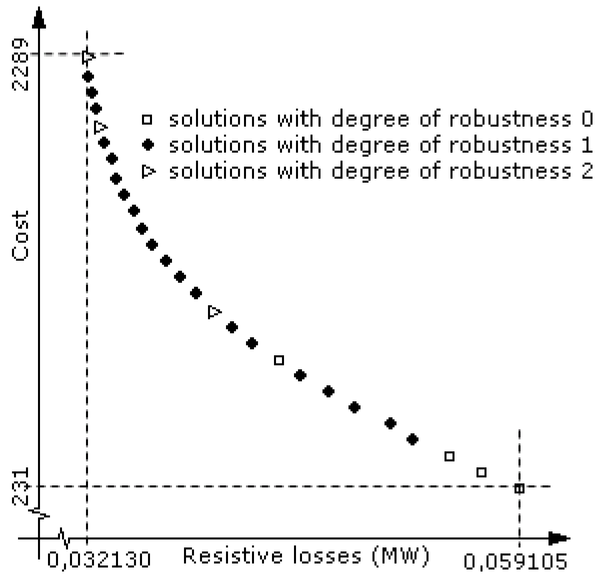

Figure 23: Pareto-front obtained with $\eta = (0.005, 200)$.

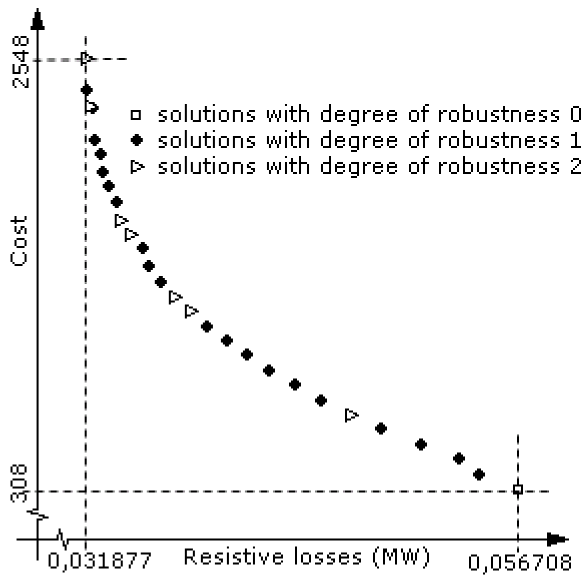

Figure 24: Pareto-front obtained with $\eta = (0.006, 250)$.

6 Conclusion

This paper presented two approaches to robustness analysis in evolutionary multi-objective optimization based on assigning a degree of robustness to solutions. These approaches address small perturbations in the values of the decision variables and the values of the objective function coefficients. The concept of degree of robustness in incorporated into the evolutionary algorithm, particularly in the computation of the fitness value of the solutions. This approach enables to classify the solutions of the Pareto-front according to the degree of robustness (solutions with different degrees of robustness can belong to the same Pareto-front), not just classifying solutions as robust or not robust (as in Deb and Gupta [11, 12]).

Information on the robustness of solutions, and not just on the objective function values, is relevant for assisting a decision maker in assessing the merit of non-dominated solutions and selecting a satisfactory compromise solution that exhibits a higher degree of stability in face of perturbations.

Research is underway to extend this approach to problems in which uncertainty is internal to the model, that is it lies on the model coefficients and it can be represented by means of interval numbers. This attempts to model coefficients as unknown but bounded without the need to specify probability distributions (as in stochastic programming) or possibility distributions (as in fuzzy programming).

References

[1] Antunes, C.H., Barrico, C., Gomes, A., Pires, D. and Martins, A. (2005). "On the Use of Evolutionary Algorithms for Reactive Power Compensation in Electrical Distribution Networks Experiments on a Case Study". Proceedings of the 6th Metaheuristics International Conference (MIC2005), 514-519. Viena, Austria, August 22-26.

[2] Barrico, C. and Antunes, C.H. (2006). "Robustness Analysis in Evolutionary Multi-Objective Optimization - with a Case Study in Electrical Distribution Networks". Presented at the II European-Latin-American Workshop on Engineering Systems (SELASI II), Porto, Portugal.

[3] Barrico, C. and Antunes, C.H. (2006). "A New Approach to Robustness Analysis in Multi-Objective Optimization". Proceedings of the 7th International Conference on Multi-Objective Programming and Goal Programming (MOPGP 2006), Loire Valley (Tours), France.

[4] Barrico, C. and Antunes, C.H. (2006). "Robustness Analysis in Multi-Objective Optimization Using a Degree of Robustness Concept". Proceedings of the 2006 IEEE World Congress on Computational Intelligence (WCCI 2006), 6778-6783.

[5] Branke, J. (1998). "Creating Robust Solutions by means of an Evolutionary Algorithm". Parallel Problem Solving from Nature vol. 5, Lecture Notes in Computer Science 1498, Springer, 119-128.

[6] Branke, J. (2000). "Efficient Evolutionary Algorithms for Searching Robust Solutions". Adaptive Computing in Design and Manufacture (ACDM 2000), 275-286.

[7] Branke, J. and Schmidh, C. (2005). "Faster Convergence by means of Fitness Estimation". In Soft Computing 9 (1), 13-20.

[8] Coello, C., Veldhuizen, D. and Lamont, G. (2002). "Evolutionary Algorithms for Solving Multi-Objective Problems". Kluwer Academic Publishers.

[9] Das, D., Nagi, H.S. and Kothari, D.P. (1994). "Novel method for solving radial distribution networks". IEE Proceedings - Generation, Transmission and Distribution, 141 (4), 291-298.

[10] Deb, K. (2001). "Multi-Objective Optimization Using Evolutionary Algorithms". John Wiley and Sons, New York.

[11] Deb, K. and Gupta, H. (2004). "Introducing Robustness in Multiple-Objective Optimization". KanGAL Report Number 2004016, Kanpur Genetic Algorithms Laboratory, Indian Institute of Technology Kanpur, India.

[12] Deb, K. and Gupta, H. (2005). "Searching for Robust Pareto-Optimal Solutions in Multi-Objective Optimization". Proceedings of the Third International Conference of Evolutionary Multi-Criteria Optimization (EMO-2005), 150-164.

[13] Deb, K., Pratap, A., Agarwal, S. and Meyarivan, T. (2002). "A fast and elitist multi-objective genetic algorithm: NSGA-II". IEEE Transactions on Evolutionary Computation, 6 (2), 182-197.

[14] Fonseca, C.M. and Fleming, P.J. (1995). "An Overview of Evolutionary Algorithms in Multiobjective Optimization". Evolutionary Computation, 3 (1), 1-16.

[15] Gomes, A., Antunes, C.H. and Martins, A. (2004). "A multiple objective evolutionary approach for the design and selection of load control strategies". IEEE Transactions on Power Systems 19 (2), 1173-1180.

[16] Hughes, E.J. (2001). "Evolutionary Multi-Objective Ranking with Uncertainty and Noise". Proceedings of the First International Conference on Evolutionary Multi-Criterion Optimization (EMO-2001), 329-343.

[17] Jin, Y. and Branke, J. (2005). "Evolutionary Optimization in Uncertain Environments - A Survey". IEEE Transactions on Evolutionary Computation, 9 (3), 1-15.

[18] Jin, Y. and Sendhoff, B. (2003). "Trade-Off between Performance and Robustness: An Evolutionary Multiobjective Approach". Proceedings of the Second International Conference on Evolutionary Multi-Criterion Optimization (EMO-2003), 237-251.

[19] Li, M., Azarm, S. and Aute, V. (2005). "A Multi-Objective Genetic Algorithm for Robust Design Optimization". Proceedings of Genetic and Evolutionary Computation Conference (GECCO'05), 771-778.

[20] Lim, D., Ong, Y.S. and Lee, B.S. (2005). "Inverse Multi-Objective Robust Evolutionary Design Optimization in the Presence of Uncertainty". Proceedings of Genetic and Evolutionary Computation Conference (GECCO'05), 55-62.

[21] Ong, Y.S., Nair, P.B. and Lum, K.Y. (2005). "Max-Min Surrogate-Assisted Evolutionary Algorithm for Robust Aerodynamic Design". IEEE Transactions on Evolutionary Computation 10 (4), 392-404.

[22] Parmee, I.C. (1996). "The Maintenance of Search Diversity for Effective Design Space Decomposition using Cluster-Oriented Genetic Algorithms (Cogas) and Multi-Agent Strategies (Gaant)". Proceedings of the Second International Conference of Adaptive Computing in Engineering and Control, 128-138.

[23] Pires, D.F., Martins, A.G. and Antunes, C.H. (2005). "A multiobjective model for VAR planning in radial distribution networks based on Tabu Search". IEEE Transactions on Power Systems 20 (2), 1089-1094.

[24] Teich, J. (2001). "Pareto-Front Exploration with Uncertain Objectives". Proceedings of the First International Conference on Evolutionary Multi-Criterion Optimization (EMO-2001), 314-328.

[25] Tsutsui, S. and Ghosh, A. (1997). "Genetic Algorithm with a Robust Solution Searching Scheme". IEEE Transactions on Evolutionary Computation, 1 (3), 201-219.