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Multicriteria Path and Tree Problems – Discussion on Exact Algorithms and Applications

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Abstract: Multicriteria/Multiobjective path and tree models are useful in many applications. Particularly, in Internet Routing Problems they seem to be very promising. In the first part of this paper we classify and present the main exact approaches dealing with several Multicriteria Path Problems putting in evidence the Shortest Path Problem. In the second part we review exact algorithms dedicated to some Multicriteria Tree Problems, namely the Minimum Spanning Tree and the Minimum Cost/Minimum Label Spanning Tree Problems. Finally, the application of these models is exemplified.

Keywords: Multiobjective path problems, Multiobjective spanning tree problems.

1 Introduction

In many applications it is potentially advantageous the use of multicriteria network models. Namely, multicriteria path and tree models are useful in transportation and communication network problems. As the authors have been involved in the development of multicriteria models/approaches dedicated to routing problems in multiservice communication networks, the applications used as examples in this work are based on cases of this area. Routing is a key functionality in a telecommunication networks, the aim of a routing model being the calculation and the selection of a sequence of network resources (corresponding to a loopless path and usually designated as route) from one origin to one or several destinations (in the case of multipath routing), as well as multicast connection (connection among a group of network nodes, corresponding to a Steiner tree problem) and broadcast (connection among all network nodes, corresponding to a spanning tree problem). These models are increasingly important as a result of the emergence of multimedia applications such as audio, video services and video conferencing. The relevant criteria for many problems in this area are not only multifaceted but commonly of an heterogeneous matter, for example economic, technical and socio-economical criteria. These issues make that in many situations the models for decision support in these areas become more realistic if different aspects are explicitly considered by building a consistent set of criteria rather than just aggregating some of them *a priori* in a single function and transforming the others into constraints as it has been done in earlier OR models

dedicated to problems of this field. Multi-Criteria Models enable deeper analysis – for instance, evaluating trade-offs among the distinct performance metrics in a mathematically consistent manner when such metrics are conflicting objectives – helping the Decision Makers (DM) looking for satisfactory compromise solution(s).

Note that even in the cases where an *a priori* aggregation of criteria is required (for instance, in online automatic decision making situations), an explicit multicriteria modeling has the advantage of enabling a deeper insight, regarding the persecution of several problem issues.

In the second section of this paper we classify and make an overview of multicriteria path problems. Taking into account its importance we pay special attention to the shortest path problem. We make an overview of exact algorithms dedicated to multicriteria path problems and we also put in evidence some relevant theoretical results.

Note that considering explicitly multiple and conflicting criteria a global optimum does not exist any more. Here the concept of optimum is substituted by the concept of efficient/non-dominated solution (a path in this case). An efficient (also known in the literature as non-inferior or Pareto optimal) path is a feasible path such that there are no other feasible paths that can improve the value of one objective function without worsening the value of at least one of the other objective functions. The mapping of an efficient path in the objective functions space lead to the corresponding non-dominated solution/point. Furthermore, these solutions can be supported and unsupported non-dominated solutions. While supported non-dominated solutions are non-dominated solutions located on the boundary of the convex hull of the feasible set in the objective functions space, unsupported non-dominated solutions are located in its interior (i.e. in the duality gaps).

Limiting the scope of the work to exact algorithms is not an hard limitation in most of the multicriteria path models because the available exact algorithmic approaches enable a very fast resolution of large instances of the problems.

In this paper we consider three categories of algorithms. Firstly, those for which there is no articulation of preferences of the DM (i.e. the aggregation of preferences is made *a posteriori*), the algorithms generate the whole set of efficient/non-dominated solutions; secondly there is a progressive articulation of preferences of the DM, the so called interactive approaches; and, finally, those where there is an *a priori* articulation of preferences, for instance, building a value/utility function.

In the interactive approaches after each calculation phase of one (or several) efficient/non-dominated solution(s), there exists a dialogue phase. This intervention of the DM is made in order to initiate a new calculation phase, and so on... The stopping condition of the process depends on the specific type of the interactive procedure.

Note that, regarding interactive procedures, the sense of our consideration of restricting this work to exact algorithms, of course, refers to the calculation phase only.

In the third section we deal with multicriteria spanning tree problems. As it is was done very

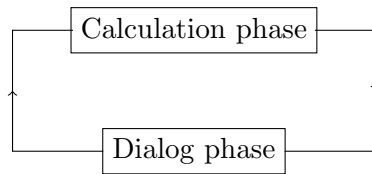


Figure 1: Interactive process

recently a review on multicriteria minimum spanning tree approaches [Ruzika and Hamacher, 2009], we limit our presentation on this subject to a critical short description of the subject. We will put more emphasis on a new and important problem, the bicriterion minimum cost/minimum label spanning tree problem. Although we recognize its great importance, in this paper we do not study the Multicriteria Steiner Tree Problem. The major reason is the inexistence of relevant exact approaches to deal with this problem.

Concerning the Multicriteria Minimum Spanning Tree Problem it must be emphasized that there are computationally efficient procedures to calculate the whole set of the supported non-dominated solutions only. On the other hand, regarding algorithms for generating the non-supported non-dominated solutions set the state of the art is very poor. So, the development of heuristics for this problem is a major issue. However, it is out of the scope of this paper.

As we told before, in the second part of this section we study the bicriterion minimum cost/minimum label spanning tree problem. This problem is also relevant in several application areas such as telecommunication networks and transportation networks. The problem involves the following: a cost value and a label (such as a color) are assigned to each arc/edge. The first criterion intends to minimize the total cost of the spanning tree (the summation of its edge costs), while the second intends to get a solution with the minimal number of different labels. Since these criteria in general are conflicting criteria, two algorithms were developed to generate the set of non-dominated spanning trees. In this paper we will discuss in some detail these approaches and we also make a short reference to other approaches where one of the criteria is considered as a constraint.

In the last section of this work we illustrate the application of some referred to algorithmic approaches, using case studies in the area of routing of modern multiservice communication networks.

2 Multiobjective path problems (MPP)

2.1 Basic concepts

Let X denote a set of feasible solutions and f^k be a function that assigns a real value to any solution in X , $k = 1, 2, \dots, r$. A multiobjective problem in X can be formulated as:

$$\min \quad f(x) = (f^1(x), f^2(x), \dots, f^r(x)) \quad (1)$$

$$\text{s. t.} \quad x \in X \quad (2)$$

Note that there is no loss of generality in considering the objective function minimization, once the maximization case can be reduced to this one. The image of the feasible set under the objective function is $f(X)$. In the following we will discuss two of these problems defined on a network, namely when X is the set of all paths between a given pair of nodes. In Section 3 we refer to the case when X is the set of spanning trees.

In general there is not a solution of the problem, that is a path from s to t , which is simultaneously optimal for all objective functions. Therefore in this context the concept of optimality is replaced by the concept of efficiency/non-dominance.

A feasible solution $x \in X$ is efficient if it does not exist any $y \in X$ with $f(y) \leq f(x)$ and $f(x) \neq f(y)$. Its image $f(x)$ is then called non-dominated. Let X_E denote the set of all efficient solutions and F_N denote the set of all non-dominated points. We distinguish two different types of efficient solutions.

- Supported efficient solutions, are efficient solutions that can be obtained as optimal solutions to a (single-objective) weighted sum problem (WSP)

$$\min_{x \in X} \left\{ \sum_{k=1}^r \lambda_k f^k(x) : \lambda_k \geq 0, k = 1, 2, \dots, r \right\}. \quad (3)$$

The set of all supported efficient solutions is denoted by X_S , while its non-dominated image is represented by F_S .

- The remaining efficient solutions in $X_{NE} = X_E - X_S$ are called non-supported solutions. They cannot be obtained as solutions solving WSPs. The set of non-supported non-dominated points is denoted by F_{NN} .

Two feasible solutions x and y are called alternative (or equivalent) if $f(x) = f(y)$. The set of all efficient solutions is called in this paper the maximal complete set, and may contain equivalent pairs of solutions. The minimal complete set is a subset of the maximal complete set that contains a single solution from any set of equivalent solutions (corresponding to a non-dominated point).

Let $(\mathcal{N}, \mathcal{A})$ be a network, with a set of n nodes, \mathcal{N} , and a set of m arcs, \mathcal{A} , together with r cost values associated with each arc (i, j) , denoted by $c_{ij}^k \in \mathbb{R}$, $k = 1, 2, \dots, r$.

Given an origin s (or initial node) and a destination t (or terminal node) in $(\mathcal{N}, \mathcal{A})$, a path from s to t in $(\mathcal{N}, \mathcal{A})$ is a sequence $p = \langle v_1, v_2, \dots, v_\ell \rangle$, where $v_1 = s$, $v_\ell = t$, $v_i \in \mathcal{N}$, $i = 1, \dots, \ell$, and $(v_i, v_{i+1}) \in \mathcal{A}$, $i = 1, \dots, \ell - 1$. For simplicity we write $i \in p$ if i is a node in the sequence p , and $(i, j) \in p$ if i immediately precedes j in p . Let \mathcal{P} denote the set of paths from s to t in $(\mathcal{N}, \mathcal{A})$, and consider the functions $f^k : \mathcal{P} \rightarrow \mathbb{R}$, for $k = 1, 2, \dots, r$. The MPP from an initial node s to a terminal node t can be formulated as

$$\min f(x) \quad (4)$$

$$\text{s. t. } p \in \mathcal{P} \quad (5)$$

or as a network flow problem:

$$\min f(x) \tag{6}$$

$$\text{s. t. } \sum_{(i,j) \in \mathcal{A}} x_{ij} - \sum_{(j,i) \in \mathcal{A}} x_{ji} = \begin{cases} 1 & \text{if } i = s \\ 0 & \text{if } i \in \mathcal{N} - \{s, t\} \\ -1 & \text{if } i = t \end{cases} \tag{7}$$

$$x_{ij} \in \{0, 1\}, \text{ for all } (i, j) \in \mathcal{A} \tag{8}$$

In the latter model x is a vector of flows on the arcs, while constraints (7) represent the flow balance at the different nodes, which can be of 1, -1 or 0, depending on the node. The arcs with one unit of flow form a path from s to t . The feasible set, X , is defined by constraints (7) and (8).

In the following we consider multiobjective problems involving the determination of paths into four groups.

The first is dedicated to the most widely known in the literature about these problems, the multiobjective shortest path problem (MSPP), where the objective functions, to be minimized, involved are of additive type (minsum), that is,

$$f^k(p) = \sum_{(i,j) \in p} c_{ij}^k, \quad k = 1, 2, \dots, r,$$

for $p \in \mathcal{P}$. Note that if $c_{ij}^k > 0$, $(i, j) \in \mathcal{A}$, $k = 1, 2, \dots, r$, then multiplicative criteria like

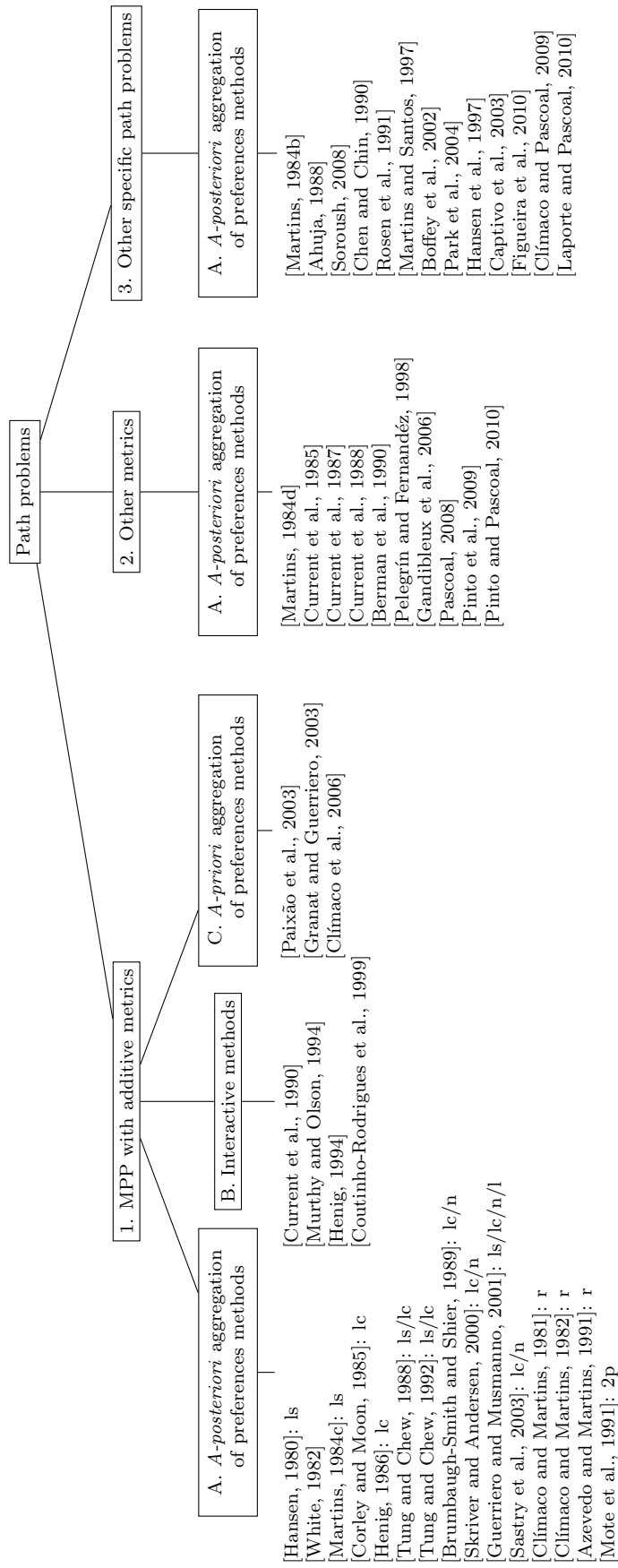
$$f^k(p) = \prod_{(i,j) \in p} c_{ij}^k,$$

can be reduced to the previous by applying logarithmic functions.

The second group comprises different types of objective combinations, the most popular of which includes additive cost functions and bottleneck type functions (minmax, or maxmin), that is, in the minmax case the aim is to minimize the functions:

$$f^k(p) = \max_{(i,j) \in p} \{c_{ij}^k\}, \quad k = 1, 2, \dots, r.$$

The third paragraph is devoted to methods dedicated to other combinatorial problems which make use of multiobjective shortest path techniques and, finally, the fourth paragraph presents a short list of path problems that deal with non-deterministic or imprecise arc parameters. These are organized into the shortest path problem in probabilistic networks, where arcs are associated with real random variables, the robust shortest path problem, where arc parameters range between two values, and the fuzzy shortest path problem, where fuzzy arc parameters are considered.



Legend: ls. label setting; lc. label correcting; n. node-selection; r. label-selection; r. ranking; 2p. two-phases.

Figure 2: Multiobjective path problem taxonomy

The works in the literature in each group are classified according to the methodology, as follows.

- A. *A posteriori* aggregation of preferences methods: Labeling, Ranking, Two phases.
- B. Interactive methods.
- C. *A priori* aggregation of preferences methods.

Figure 2 presents a taxonomy according to the described classification of the works focusing MPPs that are reviewed in the following. In the next sections a reference will be made about the main results concerning complexity bounds of polynomial time methods.

2.2 MPP with additive objective functions

The MSPP and its, most popular, particular case, the bi-objective shortest path problem (BSPP) have been studied since [Vincke, 1974], and later [Hansen, 1980] proved the problem to be intractable, presenting instances with an exponential number of efficient solutions even for the bi-objective case. Despite that, several exact methods have been developed to find the set of efficient paths. [Santos, 1999, Martins and Santos, 2000a] derived boundedness and finiteness conditions for the MSPP, if there are no negative cycles for one of the objective functions, that is cycles for which the value of at least one of the objective functions is negative, as stated below.

Theorem 1 *The MSPP is finite if and only if there are no negative cycles for at least one of the objective functions.*

Theorem 2 *The MSPP is bounded if and only if it is finite.*

Theorem 3 puts in evidence the relation between the MSPP and the determination of the efficient loopless (or simple) paths with respect to the same objective functions, equivalent under certain conditions. The proofs of these results can be found in the references above referred to.

Theorem 3 *If all the cycles in $(\mathcal{N}, \mathcal{A})$ are non-negative and for any cycle C there is some $i = 1, \dots, r$ such that $f^i(C) > 0$, then every efficient path is an efficient loopless path, and every efficient loopless path is an efficient path as well.*

The works [Ehrgott and Gandibleux, 2000, Ehrgott and Gandibleux, 2002] are multiobjective combinatorial optimization surveys that contain chapters dedicated to the MSPP, whereas classifications that include this problem are given in [Current and Min, 1986, Current and Marsh, 1993]. More recently the BSPP has been surveyed by [Skriver, 2000, Raith and Ehrgott, 2009], and the MSPP by [Tarapata, 2007]. Computational studies were presented in [Huang et al., 1996], considering two criteria, and in [Paixão and Santos, 2009] (including the use of various utility functions to define the next label to scan), considering two and more criteria. Besides reviewing the literature on the BSPP, [Raith and Ehrgott, 2009] also presents a thorough computational comparison of some

of different strategies for solving the BSPP. Namely, it tests a standard label correcting and label setting method, a ranking approach using a “near shortest path” method, that alternately identifies non-dominated supported solutions and searches within duality gaps, and the two phase method, investigating different approaches for solving problems arising in phases 1 and 2. In particular, it investigates the two phase method with ranking in phase 2. The near shortest path approach is adapted from the algorithm in [Carlyle and Wood, 2005] for enumerating all near-shortest loopless paths in a network with non-negative arc costs, that is those with a cost that does not exceed the best in more than a tolerance ϵ .

We consider three types of MSPP algorithms, generalizations of labeling techniques for the mono-objective shortest path problem, ranking methods which list paths by order of cost and eliminate solutions dominated by others, and two-phase algorithms which generate the non-dominated supported solutions of the problem and afterwards swap smaller search regions to find those that are non-dominated unsupported.

A. *A posteriori* aggregation of preferences methods (Maximal complete set computation)

- Labeling techniques

Labeling algorithms for the MSPP are generalizations of the mono-objective case. More than one objective function implies the existence of more than one path starting in s and up to each node, thus more than one label associated with that node should be used. The labels at one node may, or may not, dominate one another. These algorithms are supported by an adaptation of the Principle of Optimality for the shortest path problem, which states that every efficient path is formed by efficient subpaths. Theorem 4 (in [Santos, 1999]) states conditions for the MSPP to verify the Principle of Optimality.

Theorem 4 *If there are no negative cycles in $(\mathcal{N}, \mathcal{A})$, then the MSPP satisfies the Principle of Optimality.*

Like in the mono-objective problem, there are two types of these algorithms according to the policy for label-selection in each iteration. If the selected label always corresponds to an efficient path, then the label is permanent and we have a label setting algorithm. Otherwise, the label is temporary and we have a label correcting algorithm, one of the most popular implementations consisting of managing the set of labels to be scanned as a first in first out list (FIFO). Note that the label correcting approaches can still be classified into those that make a label correspond to a path starting in s , called label-selection methods, and, alternatively, those that associate a label with each network node and represent several paths from s to that node. In the latter case, every time a label is chosen all its paths are expanded using the network arcs. This type of approaches are known as node-selection.

In order to ensure that an efficient path is obtained the label to scan is usually picked in X by lexicographic order. The dominance test consists of comparing one label cost with the cost of the latest obtained efficient path for the bi-objective case. In fact, if labels $(a_1, b_1), \dots, (a_k, b_k)$ are selected by lexicographic order, then $a_i < a_{i+1}$, or $a_i = a_{i+1}$ and $b_i \leq b_{i+1}$ for $i = 1, \dots, k-1$. Moreover, the subsequence of that one, formed only by the non-dominated labels also satisfies $b_i \geq b_{i+1}$. Assume, without loss of generality, that some (a_k, b_k) is not dominated and let (x, y) be a label lexicographically than (a_k, b_k) . Then $a_k < x$ and $b_k \geq y$ implies that $a_i \leq x$ and $b_i \geq y$ for any i associated with the non-dominated labels subsequence. Otherwise, $a_k < x$ and $b_k < y$ implies that (x, y) is dominated. Similarly, $(a_k, b_k) = (x, y)$ implies that (x, y) is non-dominated, like the former label, whereas $a_k = x$ and $b_k < y$ means that (x, y) is dominated.

Contrarily, for the case with more than two objectives we have to check the cost of all the efficient paths already obtained. For instance, $(5, 5, 5), (10, 2, 10), (10, 5, 6)$ is a lexicographic increasing sequence of labels, where neither $(5, 5, 5)$ nor $(10, 2, 10)$ are dominated. However, the label $(10, 2, 10)$ does not dominate $(10, 5, 6)$ whereas $(5, 5, 5)$ does, so comparing $(10, 5, 6)$ with the latest label of the sequence, $(10, 2, 10)$, is not enough to decide about dominance. Finally it should be added that, similarly to the single objective problem, the arc costs should be non-negative in order to apply label setting algorithms, whereas label correcting methods are valid for any network, as long as the MSPP is finite. When labels are analysed by lexicographic order, an efficient path from s to t is obtained whenever a non-dominated label associated with node t is chosen. This means that label setting methods provide these efficient paths during their labelling phase. Otherwise, when label correcting is used the efficient paths can only be identified after all the labels are analysed.

The experiments reported in [Skriver, 2000, Huarng et al., 1996], indicate that ranking methods for the BSPP are close to an exhaustive search and thus these methods have not shown to be competitive with others. Although depending on the data structures used to implement the algorithms, [Skriver, 2000, Guerriero and Musmanno, 2001, Raith and Ehrgott, 2009] also conclude that, in general, label correcting is faster than label setting. The main reason being that label setting implies the extraction of the label with minimum weight from the set of temporary labels. This can be done either by maintaining the set of labels sorted, or else simply selecting the minimum element in the set in every extraction, but both are demanding operations, and have to be repeated several times. Still according to Raith and Ehrgott, node selection can be more advantageous than label selection because it allows to set several paths ending at node j whenever an arc (i, j) is analysed. Moreover, [Raith and Ehrgott, 2009] claim that label setting and label correcting are the methods with the best performances for most of the instances they considered and that the two phase methods can be competitive with

other approaches for the BSPP, after testing different implementation strategies for each phase, whereas the ranking approach using a “near shortest path” method performed poorly.

In the following we shortly describe some relevant references on labeling methods for the MSPP.

[Hansen, 1980]: Generalizes the single objective shortest path algorithm in [Dijkstra, 1959], concerning two objectives.

[White, 1982]: According with the author the paper discusses some of the deficiencies in the use of weighting methods to determine the efficient paths, and proposes procedures for obtaining all efficient supported paths.

[Martins, 1984c]: Generalizes Hansen’s algorithm in order to consider more than two objectives. The lexicographically smallest label is selected in the set of temporary labels, in order to ensure it corresponds to an efficient path.

[Corley and Moon, 1985]: Presents a label correcting algorithm with a label generation, similar to the single objective shortest path proposal by [Ford, 1956, Bellman, 1958].

[Henig, 1986]: Describes a dynamic programming algorithm to obtain the efficient paths. Afterwards it proposes an improvement by using a quasiconcave or a quasiconvex utility function.

[Tung and Chew, 1988, Tung and Chew, 1992]: Introduces a label setting method, that starts forward labeling with respect to the minimum sum of the two objectives and then labels the possible next steps. The visited labels are updated and marked as permanent. The second work generalizes the first for more than two objective functions.

[Brumbaugh-Smith and Shier, 1989]: The paper presents label correcting algorithms with optional rules for selecting the next label to be scanned. Implementations using different data structures are described. Managing the set of labels as a FIFO revealed to be the fastest of the tested techniques.

[Skriver and Andersen, 2000]: The work speeds up the label-correcting algorithms introduced in [Brumbaugh-Smith and Shier, 1989] by adding a pre-processing phase. A label correcting algorithm with node-selection is identified as the most successful approach for the BSPP.

[Guerriero and Musmanno, 2001]: The paper investigates label correcting and label setting methods for the multiobjective shortest path problem. It proposes new label-selection and several node-selection strategies. Computational results are presented. The paper reports instances where label-selection is superior, and others where node-selection is the best. Likewise, label setting is superior for some instances and label correcting for others.

[Sastry et al., 2003]: It detects negative cycles in networks with arbitrary arc costs, by a repeated application of a shortest path algorithm, and proposes a label correcting algorithm with node-selection similar to the one in [Brumbaugh-Smith and Shier, 1989] when the MSPP is finite. It also proposes label correcting variations of the method by [Corley and Moon, 1985]. In each iteration the labels are updated from all predecessor nodes. The algorithm halts when either none of the label sets is changed. The nodes to select are chosen randomly, and sets are updated similarly to [Yen, 1970].

- Ranking techniques

This type of algorithms use a ranking method for listing paths by non-decreasing order of one of the objective functions, which allows the partition of the set of efficient solutions into several subsets by means of a dominance test, comparing each determined path with the previous, and filtering the efficient solutions. For the bi-objective case the search starts with the lexicographic shortest path with respect to the first function, and halts when the minimal value of the other objective function is achieved. Let \mathcal{P}_{f^i} denote the set of paths in \mathcal{P} with minimal cost in terms of f^i , $i = 1, 2$, and $f_1^* = \min\{f^1(p) : p \in \mathcal{P}_{f^1}\}$, $f_2^* = \min\{f^2(p) : p \in \mathcal{P}_{f^1}\}$. Lemma 1 (in [Clímaco and Martins, 1982]) establishes a ranking stopping condition for that situation.

Lemma 1 *If p is an efficient path, then $f^i(p) \leq f_i^*$ for $i = 1, 2$.*

Let now S be the finite set of paths $S = \{p \in \mathcal{P} : f^1(p) \leq f_1^*\}$ (which can be sorted by their costs f^1) and consider the partition

$$S = \bigcup_{i=1}^k S_i, \text{ where } S_i \cap S_j = \emptyset \text{ for any } i \neq j,$$

and, given $p \in S_i$ and $q \in S_j$:

- if $i < j$ then $f^1(p) < f^1(q)$,
- if $i = j$ then $f^1(p) = f^1(q)$.

Lemma 2, [Clímaco and Martins, 1982], defines the dominance test that determines whether a path is efficient.

Lemma 2 *Path $p^* \in S_i$, for some $i \in \{2, \dots, k\}$, is efficient if and only if $f^2(p^*) = \min\{f^2(p) : p \in S_i\}$ and $f^2(q) > \min\{f^2(p) : p \in S_i\}$, for any $q \in S_j$ such that $j < i$.*

If paths are listed by non-decreasing order of f^1 , then for efficient paths f^2 are listed by non-increasing order. Therefore, the algorithm stores the candidate efficient paths in a set \mathcal{P}_X , and the dominance test for $p \in S_i$ consists of comparing $f^1(p)$ with M_1 and $f^2(p)$ with m_2 , where M_1 is the largest value of f^1 for the paths previously determined, whereas m_2 is the smallest value of f^2 for those paths. Because paths are obtained by non-decreasing order of f^1 , two situations may arise, $f^1(p) = M_1$ or $f^1(p) > M_1$. In the

first case p is dominated by some other path if $m_2 < f^2(p)$, otherwise it is candidate to be an efficient path, thus it is inserted in \mathcal{P}_X . In the second case the paths in \mathcal{P}_X are all efficient, and p is temporarily the only new efficient path candidate in \mathcal{P}_X , the only element that has been computed of a new set S_i .

For problems with more than two objectives the dominance test has to take into consideration all the cost values of the determined paths. This extension is presented in [Clímaco and Martins, 1981].

The original proposal of this type of methods uses the ranking algorithm in [Martins, 1984a]. The efficiency of the subroutine for ranking paths is crucial to the method's performance. In fact, the dominance test and the stopping condition bound the number of generated solutions, but in the worst case the computation of its complete set can require that an exponential number of paths are listed, although results in [Müller-Hannemann and Weihe, 2001] show that this limit is never reached for some concrete practical problems and the empirical results reported in the works below present reasonable running times. Shortly, ranking paths methods can be classified into three groups:

- Deletion algorithms: After the shortest path computation a new network is formed, that allows all the paths in the former, except the shortest. The procedure's repetition lists the best paths by order of cost. The first algorithm of this type was proposed in [Martins, 1984a] and was improved several times thereafter. The latest of these improvements can be found in [Martins et al., 2001]. The method described in [Jiménez and Marzal, 1999] is a recursive procedure that obtains a new path by creating the best alternative to each node of the current one. It can thus be seen as a deletion algorithm too, in particular as a recursive version of the method in [Martins et al., 2001].
- Labeling algorithms: Under the assumption that the network does not contain negative cycles the K shortest path problem satisfies an extension of the optimality principle, [Pascoal, 2000], therefore paths can be ranked by labeling methods. In order to do this a label represents some path from s up to a certain node. It is enough to store the best (at most) K labels for each network node. This type of methods were developed in [Martins et al., 2000, Guerriero et al., 2001].
- Deviation algorithms: Any path from s to t , other than the shortest, can be seen as the deviation from a shorter path, formed by three parts: an initial part common to both paths, a deviation arc and the shortest path from its head up to t . Because the shortest path from any node to t can be computed in advance by a shortest path algorithm, new candidate paths can be generated simply by selecting the next deviation arc to be used. Deviation methods are described in [Eppstein, 1998, Martins et al., 1999b, Jiménez and Marzal, 2003].

Different types of ranking methods can be applied depending on the model. Among the

most efficient algorithms for ranking loopless paths we mention [Yen, 1971, Katoh et al., 1982, Hadjiconstantinou and Christofides, 1999, Martins et al., 1999b, Martins and Pascoal, 2003, Pascoal, 2006].

The results of empirical experiments on random instances of the MSPP are reported in [Santos, 1999], involving labeling and ranking approaches. For that set of instances, concerning two or three criteria, the label correcting implementation using a FIFO outperformed both label setting and a ranking method using the deletion algorithm in [Martins and Santos, 2000b], however it mentions that for small set of tests a similar implementation but using the deviation algorithm in [Martins et al., 1999b] was more efficient than the remaining ones. [Clímaco et al., 2003] deals with a BSPP including additional resource constraints, preventing paths from exceeding a maximum bound with respect to an additive metric. In order to compare the empirical performance of the label setting method and the ranking method when applied to the BSPP, this paper reports experiments where the former is faster than the latter, using the algorithm in [Martins et al., 1999b] for the loopless paths determination. However, still according to [Clímaco et al., 2003], when resource constraints are added it is not always possible to generalize the principle of optimality. This implies to modify both algorithms, which is easier for deviation algorithms because the costs of new paths are known as they are generated, and thus non feasible paths can be discarded in an earlier phase of the algorithm.

[Clímaco and Martins, 1981, Clímaco and Martins, 1982]: The first article uses the ranking method in [Martins, 1984a] to solve the MSPP, the latter particularizes that algorithm for the case with two criteria.

[Azevedo and Martins, 1991]: The paper presents an algorithm to find the set of efficient paths in acyclic networks, based on the ranking path procedure proposed in [Martins, 1984a].

- Two-phase techniques

This algorithm works in two phases, similarly to what will be described in the context of interactive methods for the MSPP by [Current et al., 1990], and later in the paper for general multiobjective combinatorial optimization problems [Ulungu and Teghem, 1995]. In phase 1 extreme supported efficient solutions are computed, possibly taking advantage of their property of being obtainable as solutions to WSPs. In phase 2 the remaining supported and non-supported efficient solutions are computed with an enumerative approach. It is expected that the search space for the enumerative approach in phase 2 is restricted due to information obtained in phase 1 so that the associated problems can be solved quicker than by solving BSPP with a purely enumerative approach only. The search space in phase 2 can be restricted to triangles (duality gaps) given by two

consecutive supported non-dominated points.

[**Mote et al., 1991**]: Uses the unimodularity property of the constraints in the formulation of the problem as a linear program (and solve a parametric shortest path problem) to find the efficient supported paths, whereas the unsupported paths are found by a label correction algorithm.

B. Interactive methods

[**Current et al., 1990**]: This article proposes an interactive method for generating an approximation to the non-dominated solution set for two objective shortest path problems. The goal of this approach is to assist the DM in selecting the preferred or best compromise solution from among the non-dominated solutions. It works in two phases, the first of which aims at determining efficient supported solutions using a NISE-like procedure [Cohon, 1978], whereas during the second phase unsupported efficient solutions (lying inside duality gaps) are computed with the method for the constrained shortest path problem proposed in [Handler and Zang, 1980]. Phase 1 begins with the determination of the best paths with respect to each of the objective functions. If $A = (A_1, A_2)$ and $B = (B_1, B_2)$ are the objective vectors associated with those solutions, as depicted in Figure 3(a), then the WSP, with $\lambda_1 = \left| \frac{A_2 - B_2}{A_1 - B_1} \right|$ and $\lambda_2 = 1$, is solved to compute a new candidate to adjacent non-dominated solutions. Repeating this procedure and updating the weights according to the new obtained solutions until it determines the whole set of supported solutions. Phase 2 aims at identifying the unsupported solutions inside the duality gaps defined by two adjacent supported solutions, depicted in Figure 3(b), using a constrained shortest path method.

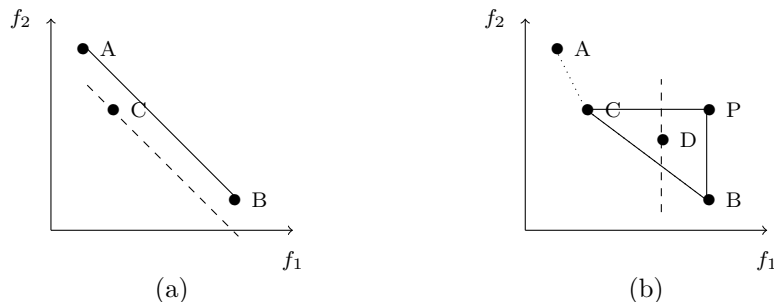


Figure 3: Non-dominated solutions of a BSPP; (a) supported solutions; (b) the first solution inside the duality gap.

[**Murthy and Olson, 1994**]: Develops an interactive procedure for the BSPP, that finds the optimal path concerning a quasi-concave and non-increasing DMs utility function. In the first stage the DM evaluates non-dominated supported path objective values. From making pairwise comparisons, the search region is reduced until it is defined by two adjacent solution costs. A labeling algorithm is then used to compute the non-supported non-dominated images,

while the domination cones resulting from pairwise comparisons by the DM shrink the search area.

[Henig, 1994]: Given an acyclic network and a preference-order relation defined through a collection of attributes, deals with when and how can the principle of optimality be combined with interactive programming. It shows that the principle of optimality is valid if and only if the preferences can be represented by a linear function of the criteria, and suggests an interactive method to assess a value function while using the principle of optimality to efficiently search for an optimal path. Along the process the DM is asked about the tradeoffs or to order solutions according to the preferences.

[Coutinho-Rodrigues et al., 1999]: This paper proposes a new interactive approach to find efficient bi-objective shortest paths based on the work [Current et al., 1990], but searching for unsupported efficient solutions (within duality gaps) using a K -shortest path procedure instead of solving the constrained problems, with better results Figure 4).

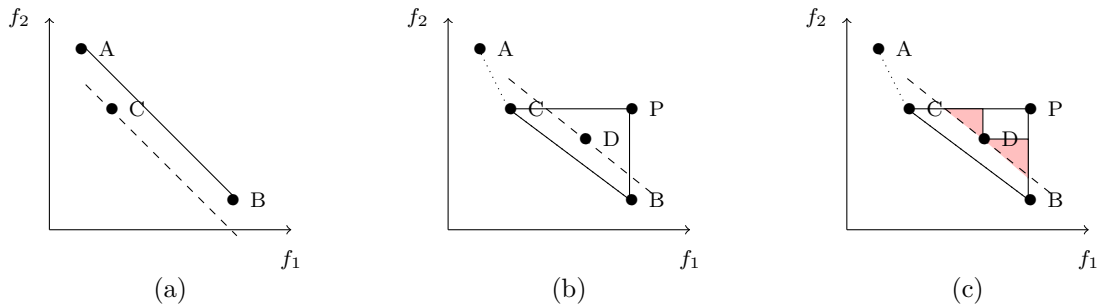


Figure 4: Non-dominated solutions of a BSPP; (a) supported solutions; (b) the first solution inside the duality gap; (c) search areas in which may exist more non-dominated solutions.

C. *A priori* aggregation of preferences methods (Maximal complete set computation)

[Paixão et al., 2003]: This work considers different metrics as utility functions. Two algorithms for solving the minimum-cost path problem (for the proposed metrics) are then proposed.

[Granat and Guerriero, 2003]: The paper suggests a labeling algorithm for finding a path that minimizes the Chebyshev distance to a given reference point.

[Clímaco et al., 2006]: Introduces a method of analysis for the automated ordering and selection of solutions of a multiobjective shortest path model. The method is based on a reference point approach, where the paths in a specific priority region are ranked by non-decreasing order of a weighted Chebyshev metric. Priority regions are defined by introduction of preference thresholds as requested and acceptable values for each objective function, Figure 5(a). In order to list paths according to this objective function a labelling algorithm is proposed. Because this problem does not satisfy the optimality principle the paper develops

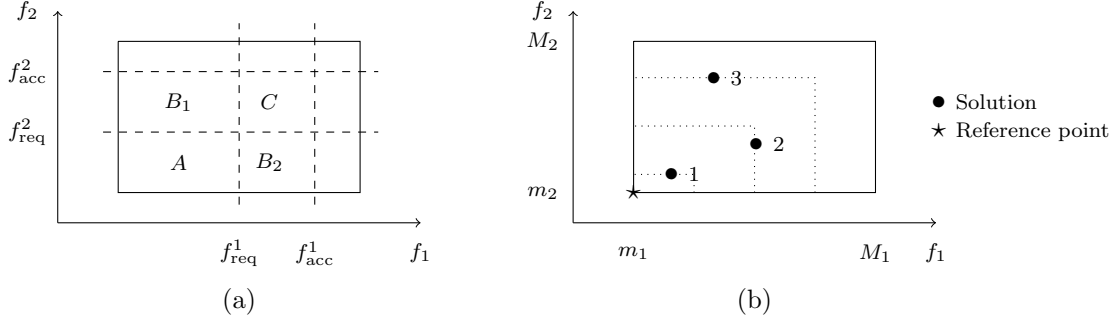


Figure 5: (a) Priority subregions; (b) Solutions ranked, in the subregion within the solid lines.

a specific dominance test to prune dominated labels, Figure 5(b). (See also the applications in Section 4.)

2.3 MPP with other objective functions

A. *A posteriori* aggregation of preferences methods

(a) Maximal complete set computation

[Martins, 1984d]: Studies bi-objective path problems with at least one maxmin function. The other can be of the same type or else a minsum or minratio. Two algorithms are presented. The first one only determines the minimal complete set of efficient paths, and the second determines the maximal complete set. Both algorithms can be used for any sort of bi-objective path problem, as long as one of the objectives is of maxmin type and an algorithm exists to determine the best path for the other objective. The minimal complete set can be determined in $\mathcal{O}(m_1 g(n))$ where m_1 is the number of bottleneck objective values and $g(n)$ denotes the number of operations to obtain an optimal solution.

[Current et al., 1985]: Introduces the maximum covering/shortest path problem and the maximum population/shortest path problem, a special case of the former model. Both models are formulated as two objective integer programs. A summary of the results of a sample problem for the latter formulation, is given. Possible modifications to, and extensions and applications of both models are also presented. With these formulations the authors extend the concept of ‘coverage’ from facility location analysis to network design and routing analysis.

[Current et al., 1987]: The authors introduce the median shortest path problem (MeSPP), a bi-objective path problem with the objectives being the minimization of the total path length and the minimization of the total travel time required for demand to reach a node on the path. An algorithm, incorporating a K shortest path algorithm

for identifying efficient solutions to the MeSPP, is presented.

[Current et al., 1988]: Introduces the minimum-covering/shortest-path (MCSP) problem, and formulates several variations of the problem. This is a two-objective path problem: minimization of the total population negatively impacted by the path and minimization of the total path length. A population is considered to be negatively impacted if the path comes within some predetermined distance of the population. Consequently, the MCSP problem extends the concept of coverage from facility location modeling to network design. Existing solution methods for the problem are discussed.

(b) Minimal complete set computation

[Berman et al., 1990]: The paper considers network optimization problems involving a general minsum objective and a maxmin measure. It contains algorithms to solve three problems considering that metrics either act as objective functions or are used to define constraints.

[Pelegrín and Fernández, 1998]: Studies the minsum-maxmin path problem and shows that the quickest path problem (QPP), reviewed in Section 2.4, is equivalent to a weighed problem associated with the former problem. It introduces an algorithm to generate some specific efficient paths of the minsum-maxmin path problem, based on [Rosen et al., 1991], as well as a solution for the QPP, coincident with the one presented in [Martins and Santos, 1997].

[Gandibleux et al., 2006]: Presents an extension of the label setting algorithm proposed by [Martins, 1984c] for the MSPP. The new version handles one more maxmin objective function, f^r , besides the minsum objective functions, f^1, \dots, f^{r-1} , and computes all the efficient paths from s to all the other nodes in the network. Given p, q two paths between the same pair of nodes, q weakly dominates p if $f^i(q) = f^i(p)$, for $i = 1, \dots, r - 1$, and $f^r(q) > f^r(p)$. The former version of the principle of optimality is not valid for this problem, because efficient paths may contain weakly efficient subpaths, therefore Gandibleux et al. propose a dominance test that makes weakly dominated labels to be stored.

[Pascoal, 2008]: Solves bi-objective path problems involving the number of arcs as one of the objective functions and another one, of minsum or maxmin type. Proposes labeling algorithms using a breadth-first search tree in order to compute the maximal and the minimal sets of efficient paths. The properties of this data structure are explored to better suit the objective function number of hops and thus simplify the labeling process. For both problems the minimal complete set can be found in time of $\mathcal{O}(mn)$, in the worst case.

[Pinto et al., 2009, Pinto and Pascoal, 2010]: Deal with a tri-objective path problem with two maxmin objectives, while the third criteria can be of any kind, as long as there exists an algorithm to compute the optimal path with respect to that func-

tion. A polynomial algorithm is presented. It generates the minimal complete set by computing the optimal solution in any subgraph where the set of arcs is restricted according to the bottleneck values. The method can be adapted to compute the maximal complete set. The latter paper enhances the method presented in the former by taking into account the objective values of the determined shortest paths to skip some of the bottleneck values, and thus reduce the number of solved shortest path problems. The same idea can still be used for other network optimization problems other than the shortest path. A labeling procedure for the problem is developed too, as well as a variant of the first method aiming at choosing the solutions with the best bottleneck value when the cost is the same. The worst case time bound of the first type algorithms is $\mathcal{O}(m_1 m_2 g(n))$, where m_1, m_2 represent the number of values that each bottleneck objective function may have and $g(n)$ is the number of operations needed to determine the single source-single destination shortest path. The labeling procedure has complexity of $\mathcal{O}(m m_1^2 m_2^2 \log(n m_1 m_2) + n m_1 m_2 \log(n m_1 m_2))$.

2.4 Other specific path problems involving two objective functions

A. *A posteriori* aggregation of preferences methods

- Minimal cost-capacity ratio path problem

[Martins, 1984b]: Presents a polynomial algorithm to determine a path from s to t , which minimizes the cost-capacity ratio,

$$f(p) = \frac{\sum_{(i,j) \in p} c_{ij}^1}{\min_{(i,j) \in p} \{c_{ij}^2\}}.$$

- Minimal cost-reliability ratio path problem

[Ahuja, 1988]: The paper deals with the minimum cost-reliability ratio path problem, that intends to determine a path p from s to t that minimizes the objective function given by

$$f(p) = \frac{\sum_{(i,j) \in p} c_{ij}^1}{\prod_{(i,j) \in p} c_{ij}^2}.$$

Based on the fact that the optimum solution of this problem is an efficient extreme solution of a bi-objective path problem, parametric programming is employed to enumerate the latter solutions and a specific sufficiency condition is used to cut down the enumeration.

- Linear fractional path problem

[Soroush, 2008]: The paper deals with the linear fractional path problem, which intends to compute a simple path p between s and t that minimizes

$$f(p) = \frac{\sum_{(i,j) \in p} c_{ij}^1}{\sum_{(i,j) \in p} c_{ij}^2}.$$

It develops an exact approach to find an optimal simple path through the network when arc attributes are non-negative. The approach uses path preference structures and elimination techniques to discard partial paths that cannot be part of an optimal path.

- Quickest path problem

Given $\sigma \in \mathbb{R}^+$, the goal of this problem is to find a path from s to t that minimizes the function

$$f(p) = \sum_{(i,j) \in p} c_{ij}^1 + \frac{\sigma}{\min_{(i,j) \in p} \{c_{ij}^2\}}.$$

The value $f(p)$ represents the total transmission time of σ data units between nodes s and t along path p if c_{ij}^1 and c_{ij}^2 denote the delay and the capacity of arc (i, j) , respectively. The QPP was introduced in [Moore, 1976], whereas a survey about this topic can be found in [Pascoal et al., 2006].

[Chen and Chin, 1990]: First notes that for constant arc capacity values, the QPP is simply the shortest path problem. The original network is enlarged by creating one subnetwork of the original, with a fixed capacity lower bound, for each of those values. The connection between those levels is made by duplicating the arcs in \mathcal{A} , starting at a level the capacity of which is higher than the arriving one. The paths of the augmented network correspond to the paths from s to t in $(\mathcal{N}, \mathcal{A})$, with a certain capacity, thus computing the shortest path from a source to any of the new terminal nodes provides a set of paths that includes the quickest one. In the worst case this algorithm runs in time of $\mathcal{O}(r(m + n \log n))$ and memory space of $\mathcal{O}(r(m + n))$, where r denotes the number of distinct arc capacity values.

[Rosen et al., 1991]: Instead of solving one shortest path problem in an enlarged network it computes several shortest paths, while ensuring that the maxmin function remains constant. This goal is achieved by considering a sequence of subnetworks of the original with fixed lower bounds of the arcs capacity. The subnetworks are obtained by deleting the arcs with a given capacity, as new shortest paths are determined. This algorithm reduces the space complexity of Chen and Chin’s algorithm to $\mathcal{O}(m + n)$, whereas the time complexity remains the same.

[Martins and Santos, 1997]: This is the first paper to study the QPP, strictly from a bi-objective point of view, and uses the algorithm for minsum-maxmin problems presented in [Martins, 1984d]. The result is very similar to the latter algorithm. However, in order to avoid the determination of dominated solutions (and thus solve fewer shortest path problems), the shortest path with a maximum capacity is found. This can be done by adapting Dijkstra’s algorithm and labelling a node when f^1 is improved, or when there is a tie in f^1 but f^2 is improved, [Martins et al., 1999a]. Because the set of arcs decreases along the sequence of networks, this algorithm (as well as Rosen et al.’s) halts

when no further paths exist in one of these networks.

[**Boffey et al., 2002**]: Notes that the set of arcs of each subnetwork in the sequence used by Rosen et al. is a subset of the previous one, and thus proposes to replace some resolutions of the shortest path problem with a simplified version of Dijkstra’s algorithm. However, the empirical tests reported do not show a substantial improvement on the algorithm’s efficiency.

[**Park et al., 2004**]: Proposes a label-setting algorithm for finding a quickest path by transforming a network to another network where an optimality property holds that each subpath of a quickest path is also a quickest path. The algorithm avoids enumerating non-dominated paths whose total transmission time is greater than the minimum.

- Minimum range and ratio path problem

[**Hansen et al., 1997**]: Studies the path problems minrange, aiming to find a path with the smallest range of arc lengths, given by

$$\max_{(i,j) \in p} \{c_{ij}^1\} - \min_{(i,j) \in p} \{c_{ij}^1\},$$

and minratio, aiming to find a path for which the ratio of the largest to the smallest arc length is minimum, given by

$$\frac{\max_{(i,j) \in p} \{c_{ij}^1\}}{\min_{(i,j) \in p} \{c_{ij}^1\}}.$$

The optimal solution of these problems is an efficient solution of the minsum-maxmin path problem, thus the algorithms enumerate candidate paths by decreasing objective value order, skipping those with an objective value greater than or equal to the best known. The introduced algorithms have $\mathcal{O}(m^2 \log \log n)$ time in directed networks and $\mathcal{O}(m^2)$ in undirected networks. The paper also investigates bi-objective extensions of the problems.

- Knapsack problem

[**Captivo et al., 2003**]: Examines the performances of a new labeling algorithm to find all the efficient paths of the bi-objective 0-1 knapsack problem, taking advantage of its conversion into a bi-objective shortest path problem over an acyclic network.

[**Figueira et al., 2010**]: Presents a generic labeling algorithm for finding all non-dominated outcomes of the multiobjective integer knapsack problem (MIKP). The algorithm is based on solving the multiple objective shortest path problem on an underlying network. Algorithms for constructing four network models, representing the MIKP, are presented.

- Disjoint path problem

[**Clímaco and Pascoal, 2009**]: Introduces a method to compute efficient bi-objective shortest pairs, each including two disjoint simple paths. The method is based on an

algorithm for ranking pairs of disjoint simple paths by non-decreasing order of cost, [Martins et al., 1999b], that is an adaptation of a path ranking algorithm applied to a network obtained from the original one after a duplication of the topology. The listed paths are then submitted to a dominance test. Each path in this new network corresponds to a pair of paths in the former one.

- Shortest path problem with relays

[Laporte and Pascoal, 2010]: Focuses the minimum cost path problem with relays (MCPFR), which consists of finding a shortest path from s to t , subject to a resource weight constraint. In order to satisfy this constraint the route nodes can be used as relays, which reset the transported weight to 0 but generates a node dependent cost. The MCPFR is modeled as a bi-objective path problem involving an aggregated function of the path and relay costs, as well as a weight function, and deals with a variant which considers all three functions separately. The authors develop labeling algorithms in which W , the bound on the weight of paths, controls the number of node labels. The labeling algorithm for the constrained single objective function version has a time complexity of $\mathcal{O}(Wm + Wn \log \max\{W, n\})$.

2.5 Path problems dealing with uncertainty

The classical path problems assume deterministic arc parameters and one or more objective functions for paths evaluation and comparison. However, in practice these values often change along time or are not known completely. As this is not the main subject of this work this section shortly refers to some of the main path problems with non-deterministic or imprecise arc parameters, and consider the shortest path problem in probabilistic networks, where each arc is associated with a real random variable, the robust shortest path problem, where each arc is associated with a real interval, and the fuzzy shortest path problem, where fuzzy arc parameters are considered.

- Stochastic shortest path problems

When the environment is stochastic and the objective function is non-linear, the optimal paths are sometimes considered as those that minimize the expected objective function. Other type of stochastic shortest path problem arises in probabilistic networks and consists of finding shortest path probability distributions in graphs whose branches have random costs.

One of the most utilized criteria to determine the optimal path is the criterion that maximizes the expected value of a utility function. This expected utility function may depend on the mean and the variance of the path distribution, and thus these two parameters act as two criteria for selecting the optimal path. This criterion is explored in [Loui, 1983, Murthy and Sarkar, 1997, Murthy and Sarkar, 1998, Rasteiro and Anjo, 2004].

- Robust shortest path problems

In robust shortest path problems with interval costs, intervals represent uncertainty about real costs and a robust path is not too far from the shortest path for each possible configuration of the arc costs.

[Dias and Clímaco, 2000] deals with partial information models for the shortest path problem. The paper proves theoretical results and proposes multicriteria algorithms based on the ranking of shortest paths to identify the efficient paths. [Guo-Shan and Yu-Hong, 2009] describes an interactive technique to decide multicriteria weights by multiple DMs in the condition of incomplete information.

Finally, the work [Hamacher et al., 2006] is not deterministic, in the sense that it considers the time-dependent BSPP, where costs are known but depend on the arrival times.

- Fuzzy shortest path problems

In practical situations, it is reasonable to assume that each arc length is a discrete fuzzy set, thus obtaining the discrete fuzzy shortest path problem. [Yu and Wei, 2007] proposes linear multiple objective programming to deal with the fuzzy shortest path problem.

3 Multiobjective spanning tree problems (MSTP)

Concerning the MSTP we limit our presentation to a critical short description of the subject (Section 3.1). In Section 3.2 we revise in more detail a new problem, the bicriterion minimum cost/minimum label spanning tree problem.

3.1 Multiobjective minimum spanning tree problems (MMSTP)

Let $(\mathcal{N}, \mathcal{A})$ be an undirected graph with \mathcal{N} being the set of n vertices and \mathcal{A} being the set of m edges. Here it is considered the existence of r criteria $(f^1(T), \dots, f^r(T))$ hence r costs c_{ij}^k , $k = 1, \dots, r$, are associated with each edge $\{i, j\}$. The criteria to be minimized are additive objectives:

$$f^k(T) = \sum_{\{i,j\} \in T} c_{ij}^k,$$

where T is a spanning tree (ST) on $(\mathcal{N}, \mathcal{A})$; or bottleneck objectives, i.e.

$$f^k(T) = \max_{\{i,j\} \in T} \{c_{ij}^k\}.$$

Considering several conflicting objectives, a global optimum does not exist. Instead, it is defined the set of efficient solutions (STs, in this case).

An efficient spanning tree is a feasible ST such that there is no other feasible one that can improve one of the objective functions without degrading the value of at least one of the others.

The mapping of an efficient solution on the objective functions space leads to the corresponding non-dominated solution.

While the supported efficient solutions can be obtained by optimizing weighted sums of the objective functions, the unsupported solutions cannot be obtained in this manner.

According with [Ruzika and Hamacher, 2009], the MMSTP approaches can be split in problems with at most one additive objective function and at least $r - 1$ bottleneck objective functions (a), and problems with at least two additive objective functions (b). Concerning (a) the authors report non-dominated set generation approaches based on enumeration procedures by [Sergienko and Perepelitsa, 1987] (a bi-objective approach) and [Melamed and Sigal, 1998] (including bi-objective and tri-objective approaches). The test instances are relatively small and this is explained by the use of enumerative procedures. As it is shown in [Ruzika and Hamacher, 2009] for problems (a) the number of non-dominated solutions is $\mathcal{O}(m^r)$, while problem (b) is NP-hard and intractable. For problems (b) the known algorithms for generating the set of efficient/non-dominated ST are dedicated to $r = 2$ and, in our opinion, the most important of them are two-phase algorithms, being capable of calculating the supported non-dominated points (or solutions), in the first phase, by running a weighted sum method a polynomial number of times. In this paper we just refer to these approaches. For more details see [Ruzika and Hamacher, 2009]. The first phase proposed by [Hamacher and Ruhe, 1994] is based on a dichotomous search scheme related to the algorithm proposed by [Cohon, 1978], in a context of linear programming (so, it is a NISE-like procedure). It is also similar to the procedure used in the interactive shortest path approach [Coutinho-Rodrigues et al., 1999], revised in Section 2. This type of procedure is used as first phase of the several two phase approaches proposed in the literature. Namely, in [Andersen et al., 1996, Ramos et al., 1998, Steiner and Radzik, 2008]. A new weighted sum based method was proposed by [Gomes da Silva and Clímaco, 2007]. It computes the set of supported non-dominated solutions of bi-objective minimum spanning tree problems (BSTP) in an ordered manner regarding the values of one of the objective functions. Although it is more elegant, it is in general slower than the method proposed by [Hamacher and Ruhe, 1994]. However, it can be useful in the development of new interactive tools as we will suggest later in this section.

Concerning the second phase, dedicated to the unsupported efficient/non-dominated solutions, the above referred to approaches propose in summary the following:

[Hamacher and Ruhe, 1994]: Proposes a neighborhood search approach enabling the calculation of feasible unsupported solutions (just guaranteeing local efficiency) and improving the coverage of the objectives space, regarding the calculation of well dispersed solutions.

[Andersen et al., 1996]: Proposes heuristics, based on the exploitation of adjacency, in order to get an approximation of the non-dominated solutions set. Of course, this procedure could not be exact, because the set of efficient solutions is not connected.

[Ramos et al., 1998]: Proposes a branch and bound approach to calculate the unsupported non-dominated solutions. It proposes a procedure to guarantee they are capable of calculating all supported and unsupported efficient trees corresponding to the non-dominated points.

[Steiner and Radzik, 2008]: Proposes the use of an r -best MST procedure. The triangles (duality gaps) between all pairs of consecutive non-dominated extreme points (corresponding to one or more efficient ST) are searched for unsupported non-dominated points using the K -best MSTs algorithm proposed by [Gabow, 1977]. The duality gaps search procedure is similar to one the proposed by [Coutinho-Rodrigues et al., 1999] dedicated to the bi-objective shortest path problem. Unfortunately this is much less efficient because the K -best MST algorithms are not capable of running in reasonable time except for small instances.

Concerning different K -best ST procedures see [Gabow, 1977, Katoh et al., 1981]. In these procedures a crucial subroutine consists of the calculation of the second best ST. In [Clímaco et al., 2008] different strategies to search the best swap of the edges that leads to the second best ST, are compared. Several fathoming conditions are derived in order to prevent useless calculations. Although some computational improvements were got, in fact they are modest and it should not be wise to foresee spectacular new improvements starting from the well known procedures.

The previously referred to MMSTP approaches are generating approaches regarding the supported non-dominated/efficient solutions and concerning the unsupported non-dominated/efficient solutions they use diversified strategies. In some cases they look for the exact calculation of the whole set of these solutions, in other cases they use approximations, sometimes just looking for some well distributed solutions. We do not discuss phase 2 in more detail because exact approaches are incapable of solving more than small ‘academic problems’, and approximate approaches are out of the scope of this paper.

In Section 2, dedicated to path problems, besides generating approaches (regarding an *a posteriori* aggregation of DM preferences), we outlined interactive approaches (regarding the progressive articulation of DM preferences) and methods involving an *a priori* articulation of preferences. Concerning MMSTPs there is no explicit reference in the scientific journals to approaches of the latter two classes, however they are subjacent to some proposals in the literature. First of all, there is a broad study on the aggregation of preferences in [Perny and Spanjaard, 2005]. It includes many possibilities, from the building of an additive cost function, to the non-dominance relation dedicated to multiobjective problems, namely exploiting preferences based on quasi-transitive binary relations defined on the solutions space.

Furthermore, concerning the *a priori* articulation of preferences, one can also refer to approaches not based on value/utility functions. For instance, MST models with $r - 1$ additional constraints, concerning $r - 1$ dimensions to be taken into account, optimizing just the remaining one. Note that the ϵ -constraint method, which is defined by the parameter $\epsilon \in \mathbb{R}$, constraining the feasible set of the single objective MST problem, may correspond to this kind of an *a priori* articulation of preferences if it is *a priori* fixed (see, for instance, [Ruzika and Hamacher, 2009] for details on the ϵ -constraint method).

Finally, some of the proposed approaches can be easily converted into interactive methods.

For instance, the first phase of the algorithm in [Hamacher and Ruhe, 1994] can easily be transformed into an interactive approach using the weighted sum method in a way similar to the one proposed by [Coutinho-Rodrigues et al., 1999] in the context of the BSPP. The algorithm in [Steiner and Radzik, 2008] can also be transformed into an interactive method by using it similarly to the full procedure proposed by Coutinho-Rodrigues et al. We believe that the development of new interactive procedures is an active area for future research. In this sense, we outline the following two phase interactive proposal dedicated to the BMSTP.

The first phase combines the use of the weighted sum methods proposed by [Hamacher and Ruhe, 1994], used similarly to the procedure used previously by Coutinho-Rodrigues et al. in the context of the BSPP, together with the method introduced by [Gomes da Silva and Clímaco, 2007], referred to above. First, an interactive version of a NISE-like approach is applied to get a well dispersed number of supported non-dominated solutions. Then, the method in [Gomes da Silva and Clímaco, 2007] is used to get some more supported non-dominated points adjacent to those previously calculated, with objective function values interesting for the DM.

In the second phase the triangles (duality gaps) where the DM is interested in looking for unsupported non-dominated solutions are searched in the following manner: Suppose the DM is interested in looking for unsupported non-dominated solutions in the triangle of Figure 6(a).

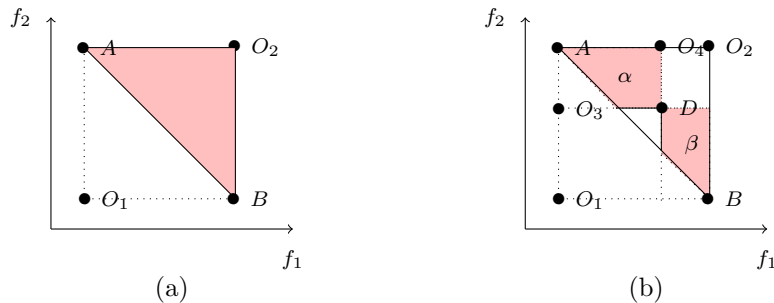


Figure 6: Computing unsupported non-dominated solutions of a bi-objective problem; (a) the duality gap; (b) search areas for eventual non-dominated solutions.

Consider O_1 as a reference point and a weighted Chebyshev metric, such that iso-distance lines to O_1 are rectangles with edges proportional to the edges of the rectangle AO_1O_2B . Using an integer programming package it is possible to get the unsupported non-dominated solution closest to O_1 in terms of the defined weighted Chebyshev metric (suppose it is point D in Figure 6(b)).

Note that the computational effort to get D is reasonable in some applications for medium size instances. For networks with 200 nodes and 800 arcs it is obtained in seconds (in the worst cases much less than 1 minute). Note that the structure of the integer programming scalar problem here solved, is very particular. So, the development of dedicated algorithms is a promising matter of research. Of course, the search can continue. Note that new eventual non-dominated solutions should be in α and/or β sub-areas of the triangle in Figure 6(b). The search in one or both sub-areas should be done by using a similar procedure, taking O_3 and O_4 as reference points, respectively.

3.2 Bi-objective minimum cost/minimum label spanning tree problem

The minimum label spanning tree problem (MLSTP) introduced in one of the seminal papers of the area by Chang and Leu [Chang and Leu, 1997], is defined as follows: given an undirected graph $(\mathcal{N}, \mathcal{A})$ with a label (color) assigned to each edge, we look for a ST of $(\mathcal{N}, \mathcal{A})$ with minimum number of different colors. This is an NP-complete problem. These authors emphasize that the purpose of constructing a ST using edges which are as similar as possible, has potential applications, for instance, in communication networks. Note that, while MLSTP ignores edge costs, [Xiong, 2005, Xiong et al., 2008] introduced a more complete model called Label Constrained Minimum Spanning Tree. The aim is to find a minimum cost ST that uses at most l labels (i.e., for instance, different types of communication media). Also in [Xiong et al., 2008], Xiong et al. proposed another evolution of the MLSTP, the Cost Constrained Minimum Label Spanning Tree problem (CCMLSTP). The above referred to problems are NP-complete and so, as expected, several authors have proposed heuristics for their resolution. For instance, Naji-Azini et al. [Naji-Azimi et al., 2010] proposed very recently a Variable Neighborhood Search method for the CCMLSTP. On the other hand, some years ago, [Cerulli et al., 2006] reporting the above referred to evolutions of MLSTP, suggested the potential usefulness of the extension of MLSTP to a model considering explicitly both criteria (BMCLST).

Recently, a first approach to this problem was proposed by [Clímaco et al., 2010].

[Clímaco et al., 2010]: In this paper, first it is described an algorithm to find the whole set of efficient STs (Maximal Set Approach). The existence of numerous STs with exactly the same cost and number of labels (so, corresponding to the same non-dominated point) lead this algorithm to perform poorly for medium size instances. Therefore a second method, enabling the computation of just the whole set of non-dominated points (Minimal Set Approach) and a corresponding efficient tree for each point was proposed.

The first algorithm is based on the following: [Clímaco and Martins, 1981] proposed a framework to find the set of efficient solutions of multicriteria problems. As we have seen in Section 2 this approach focuses on multiobjective path problems, and it was later specified by the same authors for the BSPP in [Clímaco and Martins, 1982]. The same technique can also be adapted to other multicriteria problems like the BCLST problem. With this method, solutions are ranked by non-decreasing order of one of the objective functions, while a dominance test is added in order to identify the efficient solutions set. Spanning trees can be determined by order of cost in polynomial time – see for instance [Gabow, 1977, Katoh et al., 1981]. Thus we use Gabow’s ranking algorithm together with an adaptation of the [Clímaco and Martins, 1982] dominance test to get the whole set of efficient solutions of BMCLST. For more details on this dominance test see Section 2. Note that the determination of a stopping condition to the algorithm imposes the determination of the optimum of the second objective function (i.e., the objective function not used in the ranking) and the determination of the minimal number of labels solution is an NP-hard problem. So, some polynomial time heuristics were developed in order to find approximate values. We can also refer

to the exact algorithm, but with exponential time, proposed by [Chang and Leu, 1997], as well as the mixed integer linear formulation of the MLST problem introduced by [Captivo et al., 2009]. One can also consider that the stopping condition values are defined empirically by the user, even though taking the risk of not finding the whole set of efficient solutions. As for the number of operations that this algorithm performs it depends both on the ranking methods (for complexity see [Clímaco et al., 2010]) and on the number of STs that have to be listed, which is not known in advance. However, in the general case, a factorial number of STs can occur.

It was only possible to determine the entire set of efficient trees for small size instances. The number of STs with exactly the same cost, increases very fast with the size of the instances and/or with the maximal number of labels. For instance, for random networks with 15 nodes and 50 arcs with 5 different labels, the problems run until the end, however this is not the case for the same instances considering 10 different labels.

So, it was proposed a second approach based on theoretical results detailed in [Clímaco et al., 2010]. In fact, it is sufficient to calculate the MST for the graphs with l labels – between l^* and \hat{l} , where l^* is the minimal number of labels in any ST of the instance under study and \hat{l} is the minimal number of labels corresponding to any of the best MSTs for the same instance – according to the following proposition (the proof of which can be seen in [Clímaco et al., 2010]):

Proposition 1 *There is at least an efficient tree for any l such that $l \in [l^*, \hat{l}]$, except for those $l_1 \in [l^*, \hat{l}]$, for which there exists at least a spanning tree with $l_2 < l_1$ dominating all the spanning trees with l_1 labels.*

It must be emphasized that this approach can be used avoiding to check systematically whether several STs are alternative efficient trees with the same cost and number of labels. This is advisable because otherwise the computational cost would be high and the added information would not be very valuable in most of the cases. In [Clímaco et al., 2010] several variants of this procedure are presented and tested. Namely, it is tested the way to avoid the explicit calculation of l^* , as it is computationally expensive, as we told before. Note that it is necessary to generate the whole set of sub-instances with the required number of labels. Even so, the computational results presented in the paper, using this approach, are much better than those obtained with the first procedure. For instance, problems with 500 vertices and 10 different labels are solved in less than 0.2 seconds. The computational burden increases for 20 label problems, but, even so, those problems are solved in reasonable time for off-line applications. So it seems that this second procedure can be useful, namely in telecommunication applications.

4 Applications

In this section we make a short review of some significant applications of several algorithms previously presented in the framework of routing problems in multiservice modern telecommunication networks.

- **[Clímaco et al., 2003]:** Presents an application of a generating algorithm dedicated to the BSPP. In communication networks routing problems supporting multiple services, namely, multimedia applications, the objective functions are concerned with the necessity of minimizing the consumption of transmission resources along a path and to obtain a minimum negative impact on all other traffic flows that may use the network. The specific models of these cost functions and of the QoS constraints depend on the type of multimedia service associated with the “calls” which are being sorted from origin to destination. Typical objective functions are the number of arcs (usually designated in telecommunications as hops or links) and the cost of accepting a call in each arc, measured by an appropriate traffic model related to the bandwidth available in each link, typically the minimum bandwidth required by the call and the maximum allowed delay and jitter.

Although classical models in this area are mono-objective, in many situations it is important to consider different, eventually conflicting objectives. Having in mind to explore the multi-dimensional nature of this type of problem this paper considers a bicriterion model dedicated to calculating the whole set of efficient paths for traffic flows associated with multimedia type services in multiservice networks. For this purpose, an exact algorithmic approach was developed based on the BSPP algorithm by Clímaco and Martins [Clímaco and Martins, 1982] and the MPS algorithm [Martins et al., 1999b].

The speed of the proposed approach in calculating the set of efficient/non-dominated solutions seems to make it rather appropriate for application in real world networks. Note that, it would be expectable that the use of a labeling algorithm could be a better approach, however, the explicit consideration of additional constraints in the BSPP justifies the better performance of the ranking algorithm. This new approach was tested with success on randomly generated networks as well as on US intercity based networks, simulating realistic types of applications. To show the applicability and performance of the algorithm, a specific model of application to a routing problem of video traffic in Asynchronous Transfer Mode (ATM) type networks was developed and presented in [Pornaivalai et al., 1998]. Test instances with up to 3000 nodes and an average of node degree of 4 were tested. They ran in relatively short processing times (in a few seconds) and with modest memory requirements. Note that an average node degree 4 is close to the typical average node degree of many current real world networks of this type.

In the above referred to paper it is not proposed a procedure to make the choice among efficient paths. In many situations in real world applications this can be done by the introduction of preference thresholds defining regions with different priorities, usually concerning the objective functions space (Figure 5). A first priority region is defined by the points for which both requested (or aspiration) values are satisfied. Second priority regions are such that only one of the requested values is met and the acceptable (or reservation) value for the other metric is also guaranteed. A further distinction can be made between these second

priority regions by establishing a preference order on the objective functions. A third priority region can be considered where only acceptable values for both metrics are fulfilled. This was used in the model proposed in the next paragraph.

- **[Clímaco et al., 2004]:** In many situations a very fast automatic choice is required. For these situations it was proposed an approach based on the definition of weights for each objective function and the calculation of K -shortest paths using the weighted sum as single objective function, together with the priority regions previously referred to. It is well known in multicriteria analysis that fixing weights, in this type of approaches, is a difficult task. Nevertheless in the context of this bicriteria routing model the engineering framework of the problem makes the fixation of weights a potentially consistent approach. For details on this subject and for computational tests see [Clímaco et al., 2004]. Note that the computational required times for the size of networks previously referred to is of the tens of milliseconds order, which makes this algorithmic approach suitable for application in real-time automated routing calculation procedure.
- **[Clímaco et al., 2006]:** Alternatively, it was proposed a method based on a reference point approach, where the paths in a specific priority region are ranked by non-decreasing order of a weighted Chebyshev metric. The routine proposed for ranking paths according to the Chebyshev metric consists of a labeling algorithm, which has the advantage of enabling a simple incorporation of the verification of additional constraints on the solutions. The method for ordering and selecting solutions uses preference thresholds for the objective functions as specified above. A key feature of the method is the appropriate choice of reference points and the tuning of the weights concerning the definition of an appropriate weighted Chebyshev metric allowing an exhaustive search in each sub-region, searching the different sub-regions taking into account their priority order. This approach was tested on a video-traffic routing model and several loopless paths were ranked in, at most, 24 milliseconds, on a network with 2000 nodes (and average node degree 4).
- **[Gomes et al., 2009]:** More recently, motivated by a project with Portuguese Telecom Innovation, it was developed a bicriterion model for obtaining a topological path (unidirectional or symmetric bidirectional) for each lightpath request in an all-optical wavelength routed WDM network. The solution approach here proposed is closely related to those referred to above.
- **[Craveirinha et al., 2009]:** Finally, we summarize a recent work regarding a MMSTP. The Multiprotocol Label Switching (MPLS) platform for Internet enables the implementation of advanced routing schemes, namely explicit routes satisfying QoS requirements and is prepared for dealing with multipath routing, including multicast and broadcast connections. This work addressed a new BMSTP model intended for routing broadcast messages in MPLS networks or constructing tree-based overlay networks. The aim of the model is to obtain

STs that are compromise solutions with respect to two important traffic engineering metrics: load balancing cost and average delay bound. A set of STs corresponding to the full set of supported non-dominated points is calculated and a reference point like approach similar to the one in [Clímaco et al., 2006] is used to select a solution in a filtered subset, obtained from a maximal delay bound path constraint. An application model and a set of experiments on randomly generated Internet type topologies was also presented.

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