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Some thoughts about the simultaneous location of franchising services with preferential rights

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Abstract

In this research report we will describe and study a competitive discrete location problem where two decision-makers (players) will have to decide where to locate their own facilities. We will focus on the situation in which the players must decide simultaneously, unsure about the decisions of one another. Most problems described in the literature consider sequential rather than simultaneous decisions. As a matter of fact, if there is only one decision-maker that will locate multiple facilities, and no comparable facilities exist in the market, it is rather indifferent to consider the location decisions as being made sequentially or simultaneously. In a competitive environment, most problems consider that there is a set of known and already located facilities, and new facilities will have to be located, competing with the existing ones. In the presence of more than one decision-maker, most problems found in the literature belong to the class of Stackelberg location problems, where one decision-maker, the leader, locates first and then the other decision-maker, the follower, locates second, knowing the decisions made by the first. These types of problems are sequential problems and differ significantly from the problem tackled in this research report, where we explicitly consider simultaneous, non-cooperative discrete location decisions. We will describe the problem studied, propose a mathematical formulation and describe some of the problem's properties.

Keywords: location; game theory; simultaneous decisions; Nash equilibrium.

1 Introduction

Competitive location problems consider the situation where it is not sufficient for a given decision-maker to consider only his own facilities when faced with a location decision. Most of the times, these facilities will compete with similar facilities in the market, so that the customers' share that will be assigned to the decision-maker's facilities depends on his own choices as much as on the competitors' decisions. Sometimes the location decision can also be linked to other related decisions, like at what price and in which quantities to make the commodities available.

Competitive location problems have attracted the attention of researchers of different fields of research (economy, mathematical programming, location science, game theory) for many years, have been studied in different forms and from different points of view. The published work considered by most authors as the first introducing competition in location models is Hotelling, 1929. The Hotelling model, as it is now known, considers the problem of two competitors locating one facility each, along a line where customers are uniformly distributed, having as objective the maximization of the demand captured.

Among the competitive location linear programming problems, most approaches either consider that the firms already present in the market will not be able to react to the decision-maker's new chosen locations or consider a Stackelberg problem, where there is a follower that will react to a leader, knowing what the leader has decided. For a review of competitive location problems see, for instance, Eiselt et al., 1993, Eiselt and Laporte, 1996, Plastria, 2001.

In this paper we are interested in discrete location problems, where two non-cooperative decision-makers will have to decide where to locate their facilities. We present both the sequential and the simultaneous problems, but we focus on the latter, where each decision-maker must make his decisions without knowing the decisions the other one will make. This differentiates this problem from the Stackelberg location problems, where there are two decision-makers that make decisions in a sequential manner: the leader decides first and then the follower will have perfect knowledge of the locations chosen by the leader when making his own decision. Stackelberg location problems can be formulated as bi-level linear programming problems, and have as optimal solution a Nash equilibrium solution with pure strategies.

We approach this simultaneous decision problem from a mathematical programming point of view and from a game theory point of view. As a matter of fact, this problem can be seen as a full information game (because each player knows the payoffs and strategies of the other), with a finite number of players (the two decision-makers), and a finite number of pure strategies (for each player, a pure strategy can be defined as a particular combination of locations, out of the set of potential new locations, where the player has chosen to open facilities). It is known that these games have a Nash equilibrium, possibly with mixed strategies. The solutions that constitute a Nash equilibrium will be considered as the admissible solutions to our problem.

In the next section we will describe the problems studied and put them into proper context, showing some mathematical formulations. In section 3 we will present some properties of the problem. Finally, in section 4 we will present some concluding remarks and paths for future research.

2 Describing the problem

Consider a situation where a franchiser intends to open new facilities in a given area. There are two potential interested investors, and the facilities to be open will compete among themselves. They provide the same type of commodities to consumers, at the same prices, and it is assumed that customers patronize the closest available facility. The franchiser defines the finite set of potential locations for facilities, but he is not familiar with the demand patterns of the area. So, he will define more potential locations than he expects the investors to choose, and leave the choices among them to the investors, who are better acquainted with the area. The franchiser payoff will be a percentage calculated over the total demand assigned to the new facilities. Each investor is interested in maximizing the total demand that is assigned to his own facilities[1]. Each investor is aware of the fixed costs incurred by opening each and every facility, which can be different for both investors. Each investor has a budget constraint. They are also aware of the demand associated with each customer. This demand will not increase with distance, meaning that the closest the assigned facility is of a given customer, the greatest the demand from the customer.

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[1] In fact, it will usually make sense to consider that the payoff of both the franchisees and the franchiser is given by their profit instead of the demand. We use the expression demand to allow a wider range of applications. Notice that, if we consider that the demand is given in terms of sales value and the profit is a constant percentage of the sales, then maximizing the demand is the same as maximizing the profit.

This problem can also be interpreted as a game, with two players and a finite set of strategies for each player. That is why we will not distinguish between investor, decision-maker and player, and will use these terms interchangeably as having the same meaning.

We will now describe two different situations. First we will consider the sequential problem, and then we will introduce the simultaneous decision problem.

2.1 Sequential decisions

Consider that one of the potential investors is already a franchisee, and has preference rights over the available facility locations. This means that these potential locations are offered to this investor first, and only if he is not interested are they offered to the second investor. The first investor will be referred to as the leader and the second will be called the follower. This problem can be seen as a discrete Stackelberg location problem, and can be formulated as a bi-level programming problem.

Consider the following definitions:

F - set of pre-existing facilities that belong to investor 1 (leader);

G - set of potential locations for new facilities;

J - set of customers;

d_{ij} - demand associated with customer *j* when he is assigned to a facility located at *i*;

c_{ij} - distance between customer *j* and location *i*;

f_{ip} - fixed cost associated with investor *p* opening a facility at location *i* (and such that $f_{ip} = 0, \forall i \in F$)

α_p - percentage over the demand captured to be paid to the franchiser by investor *p*;

O_p - maximum budget available to investor *p*;

We defined before that demand will not increase with distance. We will additionally assume that potential locations at the same distance will capture the same demand. This means that:

$$c_{ij} \leq c_{kj} \Rightarrow d_{ij} \geq d_{kj}, \forall i,k \in F \cup G, \forall j \in J \qquad (2\text{-}1)$$

$$c_{ij} = c_{kj} \Rightarrow d_{ij} = d_{kj}, \forall i,k \in F \cup G, \forall j \in J \qquad (2\text{-}2)$$

Let us define the following decision variables:

$$y_i = \begin{cases} 1, & \text{if investor 1 either opens a facility at } i \text{ or has a pre-existing} \\ & \text{facility at } i \\ 0, & \text{otherwise} \end{cases} \quad , \forall i \in F \cup G;$$

$$z_i = \begin{cases} 1, & \text{if investor 2 opens a facility at } i \\ 0, & \text{otherwise} \end{cases} \quad , \forall i \in F \cup G;$$

$$x_{ij} = \begin{cases} 1, & \text{if client } j \text{ is assigned to facility } i \text{ that belongs to investor 1} \\ 0, & \text{otherwise} \end{cases} \quad , \forall i \in F \cup G, \forall j \in J;$$

$$m_{ij} = \begin{cases} 1, & \text{if client } j \text{ is assigned to facility } i \text{ that belongs to investor 2} \\ 0, & \text{otherwise} \end{cases} \quad , \forall i \in F \cup G, \forall j \in J.$$

We can then formulate the sequential location problem as follows:

Formulation A

$$Max \sum_{i \in F \cup G} \sum_{j \in J} (1-\alpha_1) d_{ij} x_{ij} \tag{2-3}$$

Subject to:

$$\sum_{i \in G} f_{i1} y_i \leq O_1 \tag{2-4}$$

$$y_i = 1, \forall i \in F \tag{2-5}$$

$$Max \sum_{i \in G} \sum_{j \in J} (1-\alpha_2) d_{ij} m_{ij} \tag{2-6}$$

Subject to:

$$y_i + z_i \leq 1, \forall i \in F \cup G \tag{2-7}$$

$$\sum_{i \in G} f_{i2} z_i \leq O_2 \tag{2-8}$$

$$Min \sum_i \sum_j c_{ij} \left(x_{ij} + m_{ij} \right) \tag{2-9}$$

Subject to:

$$m_{ij} \leq z_i, \forall i \in F \cup G, j \in J \tag{2-10}$$

$$x_{ij} \leq y_i, \forall i \in F \cup G, j \in J \tag{2-11}$$

$$\sum_{i \in F \cup G} (x_{ij} + m_{ij}) = 1, \forall j \in J \tag{2-12}$$

$$z_i, y_i, x_{ij}, m_{ij} \in \{0,1\}$$

At the first stage we consider the decision made by the leader, investor 1. The objective function maximizes the total demand assigned, considering the percentage that has to be paid to the franchiser. Restriction (2-4) represents the budget constraint. The leader will have to consider the best decision made by the follower (second stage), who will also want to maximize the assigned demand, subject to a budget constraint. The set of restrictions (2-7) guarantee that the follower will only be able to open facilities where the leader has not. After both decision-makers having decided where to locate their own facilities, customers will choose the facilities to be assigned to (third stage). The customers will patronize the closest open facility, so the objective function considers the minimization of the total distance. Customers will only be assigned to open facilities (restrictions (2-10) and (2-11)) and every customer will be assigned to exactly one facility (restrictions (2-12)).

The third stage of this three-level programming problem can be easily replaced by restrictions that will guarantee that each customer will be assigned to the closest facility. In order to do so, we define a set T_{ij} that represents the potential locations that are closest to customer *j* than location *i*. Formally:

$$T_{ij} = \left\{ k \in F \cup G : c_{kj} < c_{ij} \right\}, \forall i \in F \cup G, \forall j \in J \ .$$

For the sake of simplicity, let us assume for now that for each and every customer *j* there are not two potential facility locations at the same distance from him. A two-stage formulation that is equivalent to Formulation A can be defined as:

Formulation B

$$Max \sum_{i\in F\cup G} \sum_{j\in J} (1-\alpha_1) d_{ij} x_{ij} \qquad (\ 2\text{-}3\ )$$

Subject to:

$$\sum_{i\in G} f_{i1} y_i \le O_1 \qquad (\ 2\text{-}4\ )$$

$$y_i = 1, \forall i \in F \qquad (\ 2\text{-}5\ )$$

$$Max \sum_{i\in G} \sum_{j\in J} (1-\alpha_2) d_{ij} m_{ij} \qquad (\ 2\text{-}6\ )$$

Subject to:

$$y_i + z_i \le 1, \forall i \in F \cup G \qquad (\ 2\text{-}7\ )$$

$$\sum_{i\in G} f_{i2} z_i \le O_2 \qquad (\ 2\text{-}8\ )$$

$$m_{ij} \le z_i, \forall i \in F \cup G, j \in J \qquad (\ 2\text{-}10\ )$$

$$x_{ij} \le y_i, \forall i \in F \cup G, j \in J \qquad (\ 2\text{-}11\ )$$

$$\sum_{i\in F\cup G} (x_{ij} + m_{ij}) = 1, \forall j \in J \qquad (\ 2\text{-}12\ )$$

$$m_{ij} + x_{ij} \le 1 - z_k - y_k, \forall i \in F \cup G, j \in J, k \in T_{ij} \qquad (\ 2\text{-}13\ )$$

$$z_i, y_i \in \{0,1\}, \forall i \in G$$
$$x_{ij} \in \{0,1\}, \forall i \in F \cup G, \forall j \in J$$
$$m_{ij} \in \{0,1\}, \forall i \in F \cup G, \forall j \in J$$

The set of restrictions (2-13) guarantees that each customer will only be assigned to the closest open facility.

This is a bi-level programming formulation, and can be solved using known algorithms (for instance, Bard and Moore, 1992). It is interesting to note that the optimal solution to this bi-level programming problem is no more than a Nash equilibrium with pure strategies: the follower will always consider his best response to the actions taken by the leader, that in turn will make his optimal decision by assuming this “best response” behavior of the follower and choosing a best response decision also.

2.2 Simultaneous decisions

Let us now suppose that the potential locations for facilities are simultaneously offered to both investors and that they will have to decide simultaneously whether or not they are interested in each of the locations. In this situation, it will be necessary to clarify what happens if both investors apply for the same location. The franchiser will have to decide what to do, and he can choose one the following approaches:

- He patronizes one investor, in detriment of the other. For the sake of simplicity, consider that the franchiser chooses investor 1. This means that if both apply for the same location, then the franchiser will allow investor 1 to open the facility, and investor 2 will not be able to do so. In this situation it is assumed that at most one facility can be open at each location.
- The franchiser will let both investors open facilities, meaning that it will be possible to have two facilities open at the same location, each belonging to one of the investors.

Whichever is the approach followed by the franchiser, we assume that the rules are known a priori by both investors. In terms of the definition and the behavior of the problem, these two approaches originate two completely different simultaneous decision problems. To neither of them can bi-level programming formulations be applied. As a matter of fact, we are not aware of any consistent way of formulating these simultaneous decision problems as linear programming problems, despite the fact that they are intrinsically linear. The problem raised by the second option is described and tackled in Godinho and Dias, 2010. We will next consider the problem originated by the first option.

Let us assume that investor 1 has preference rights, and whenever both investors choose the same location, investor 1 will open the facility in detriment of investor 2.

Consider the following new set of variables:

$$w_i = \begin{cases} 1, & \text{if investor 2 bids for opening a facility at } i \\ 0, & \text{otherwise} \end{cases}, \forall i \in F \cup G .$$

This new set of variables will allow the distinction between the two situations: of *bidding for* and of *being able to open* a facility in a given location. This distinction is crucial for investor 2.

It is also important to define whether or not investor 2’s bids can exceed his budget. In fact, in some cases investor 2 will know beforehand that he will not be able to open facilities in all of his chosen locations, because some locations will be chosen by investor 1. In such cases, it may be rational for him to bid for more locations than he can pay, if he knows that he will only win a subset of locations that fits his budget. As an example, consider the case in which there are only two available locations, with equal investment costs, and the budget of each investor allows each of them to open one facility only. It may be the case that investor 2 does not know which location investor 1 will choose. However, investor 2 knows that investor 1 will choose a location, so he may bid for both locations in order to ensure that he will keep the location that is not chosen by investor 1.

We will therefore consider two different cases:

1) Investor 2 bids must not exceed his budget;
2) Investor 2 may bid for more locations than he can pay, as long as he can be sure that he will only be given a set of locations that he can afford.

In both cases, we are considering that investor 2 will have to open all the facilities that he has bid and investor 1 has not. In either case there is not a straightforward way to represent the simultaneous discrete location decision problem as a linear program, basically due to the fact that there are two decision-makers that will control different sets of variables. There are restrictions that have to be considered by each decision-maker separately, and then there are restrictions responsible for connecting the decisions of them both. Putting it in another way:

- Investor 1 controls variables $y_i, \forall i \in G$;
- Investor 2 controls variables $w_i, \forall i \in G$;
- The franchiser controls variables $z_i, \forall i \in G$;
- Customers control variables $x_{ij}, m_{ij} \forall i \in F \cup G, j \in J$.

The franchiser will decide what are the values of variables $z_i, \forall i \in G$ according to the rules of the game known *a priori*. The same happens with customers: variables $x_{ij}, m_{ij} \forall i \in F \cup G, j \in J$ will be set according to known rules (in this case, the minimum distance criteria). Decisions taken by the franchiser and by the customers are not controlled by the two investors, despite the fact that these decisions will also determine each of the investors' payoff. However, both investors know the rules that will determine the variables' values. As a matter of fact, if variables y_i and w_i are fixed, then the corresponding values for x_{ij}, m_{ij} and z_i can be computed.

Each investor will make his decisions (that is, choose the values for the variables he controls) conditioned only by his own constraints. We will then have a set of connection restrictions that will define the values of the remaining variables (the ones that are not directly controlled by the investor), according to the rules.

We now formulate the problem corresponding to case 1, following the representation for simultaneous decisions used by Godinho and Dias, 2010.

Formulation S1

Decision-Maker 1

$$Max \sum_{i \in F \cup G} \sum_{j \in J} (1 - \alpha_1) d_{ij} x_{ij} \tag{ 2-3 }$$

Subject to:

$$\sum_{i \in G} f_{i1} y_i \le O_1 \tag{ 2-4 }$$

$$y_i = 1, \forall i \in F \tag{ 2-5 }$$

Decision-Maker 2

$$Max \sum_{i \in F \cup G} \sum_{j \in J} (1 - \alpha_2) d_{ij} m_{ij} \tag{ 2-6 }$$

Subject to:

$$\sum_{i \in G} f_{i2} w_i \le O_2 \tag{ 2-14 }$$

$$w_i = 0, \forall i \in F \tag{ 2-15 }$$

Connection restrictions

$$y_i + z_i \le 1, \forall i \in F \cup G \tag{ 2-7 }$$

$$m_{ij} \le z_i, \forall i \in F \cup G, j \in J \tag{ 2-10 }$$

$$x_{ij} \le y_i, \forall i \in F \cup G, j \in J \tag{ 2-11 }$$

$$\sum_{i \in F \cup G} (x_{ij} + m_{ij}) = 1, \forall j \in J \tag{ 2-12 }$$

$$m_{ij} + x_{ij} \le 1 - z_k - y_k, \forall i \in G, j \in J, k \in T_{ij} \tag{ 2-13 }$$

$$z_i \le w_i, \forall i \in F \cup G \tag{ 2-16 }$$

$$z_i \ge w_i - y_i, \forall i \in F \cup G \tag{ 2-17 }$$

$$z_i, w_i, y_i \in \{0,1\}, \forall i \in F \cup G$$

$$x_{ij} \in \{0,1\}, \forall i \in F \cup G, j \in J$$

$$m_{ij} \in \{0,1\}, \forall i \in F \cup G, j \in J$$

Restrictions (2-16) guarantee that investor 2 will only be able to open a facility at *i* if he has shown interest in it. Restrictions (2-17) guarantee that if investor 2 has shown interest in location *i*, and investor 1 has not, then investor 2 will have to open a facility at *i*.

Let us now consider case 2. If we denote by M an arbitrarily large number and by ε an arbitrarily small number, with $M, \varepsilon > 0$, we may use the following formulation:

Formulation S2

Decision-Maker 1

$$Max \sum_{i \in F \cup G} \sum_{j \in J} (1-\alpha_1) d_{ij} x_{ij} \tag{2-3}$$

Subject to:

$$\sum_{i \in G} f_{i1} y_i \le O_1 \tag{2-4}$$

$$y_i = 1, \forall i \in F \tag{2-5}$$

Decision-Maker 2

$$Max \sum_{i \in F \cup G} \sum_{j \in J} (1-\alpha_2) d_{ij} m_{ij} - Mv \tag{2-18}$$

Subject to:

$$w_i = 0, \forall i \in F \tag{2-19}$$

Connection restrictions

$$y_i + z_i \le 1, \forall i \in F \cup G \tag{2-7}$$

$$m_{ij} \le z_i, \forall i \in F \cup G, j \in J \tag{2-10}$$

$$x_{ij} \le y_i, \forall i \in F \cup G, j \in J \tag{2-11}$$

$$\sum_{i \in F \cup G} (x_{ij} + m_{ij}) = 1, \forall j \in J \tag{2-12}$$

$$m_{ij} + x_{ij} \le 1 - z_k - y_k, \forall i \in G, j \in J, k \in T_{ij} \tag{2-13}$$

$$z_i \le w_i, \forall i \in F \cup G \tag{2-16}$$

$$z_i \ge w_i - y_i, \forall i \in F \cup G \tag{2-20}$$

$$\sum_{i \in G} f_{i2} z_i \le O_2 + Mv \tag{2-21}$$

$$\sum_{i \in G} f_{i2} z_i \ge O_2 + M(v-1) + \varepsilon \tag{2-22}$$

$$z_i, w_i, y_i, v \in \{0,1\}, \forall i \in F \cup G$$

$$x_{ij} \in \{0,1\}, \forall i \in F \cup G, j \in J$$

$$m_{ij} \in \{0,1\}, \forall i \in F \cup G, j \in J$$

In this case, we are allowing investor 2 to choose more locations than the ones he can afford. Investor 2 has no longer to guarantee the satisfiability of restriction (2-14). Nevertheless, as he is obliged to open all facilities *i* such that $w_i = 1$ and $y_i = 0$, we introduce an arbitrarily large penalty for the case in which this investor will exceed his budget considering the opening of the facilities. This ensures that he will choose a set of locations such that he can afford the ones that will be given to him.

Variable v is not controlled by investor 2. We can think of this variable as being controlled by the franchiser, who wants to guarantee trustworthy bids from both players. The value of this variable will be defined by the following rule: if investor 2's budget can accommodate the opening of all the facilities given to him by the franchiser, then $v = 0$; otherwise $v = 1$. Constraint (2-22) guarantees that this rule is followed. It is possible to set $v = 1$ if and only if $\sum_{i \in G} f_{i2} z_i > O_2$.

If we want to consider the situation where it is possible that a customer is equally distant from two or more open facilities, we will then resort to an auxiliary set T'_j which includes the pairs (i,k) of potential locations for facilities such that *i* is as far from *j* as *k*. Formally:

$$T'_j = \{(i,k) : i,k \in F \cup G \wedge c_{ij} = c_{kj}\}, \forall j \in J$$

By defining T'_j, restrictions (2-13) can be replaced by (2-21) and (2-22).

$$\begin{aligned}
&m_{ij} \le 1 - z_k, \forall i \in G, j \in J, k \in T_{ij} \\
&m_{ij} \le 1 - y_k, \forall i \in G, j \in J, k \in T_{ij} \\
&x_{ij} \le 1 - z_k, \forall i \in F \cup G, j \in J, k \in T_{ij} \\
&x_{ij} \le 1 - y_k, \forall i \in F \cup G, j \in J, k \in T_{ij}
\end{aligned} \qquad (\text{ 2-21 })$$

$$\begin{aligned}
&y_i + y_k - 2 \le x_{ij} - x_{kj} \le 2 - y_i - y_k, \forall j \in J, (i,k) \in T'_j \\
&z_i + z_k - 2 \le m_{ij} - m_{kj} \le 2 - z_i - z_k, \forall j \in J, (i,k) \in T'_j \\
&y_i + z_k - 2 \le x_{ij} - m_{kj} \le 2 - y_i - z_k, \forall j \in J, (i,k) \in T'_j
\end{aligned} \qquad (\text{ 2-22 })$$

The assignment variables will then change their behavior:

$x_{ij} \in [0,1], \forall i \in G \cup F, j \in J$

$m_{ij} \in [0,1], \forall i \in G \cup F, j \in J$

Each solution to this problem is composed by a set of y_i variables' values, which we will denote as vector y and a set of w_i variables' values, which we will denote as vector w. It is interesting to define what are considered admissible solutions to this problem. As a matter of fact, interpreting this problem as a game, y is a strategy for player 1, and w is a strategy for player 2. We will consider as admissible solutions that are Nash Equilibrium solutions, with pure or mixed strategies. A solution (y, w) is admissible, if y is a best response to w and vice-versa. Under this view, (2-3), (2-6) and (2-18) are in fact restrictions to the problem. So we are really in the presence of an admissibility problem, rather than an optimization problem, in the sense that we want to find admissible solutions instead of optimal solutions.

As this game has a finite number of players and a finite number of strategies, we can be sure that at least one Nash equilibrium solution exists, possibly with mixed strategies.

3 Properties of the problem

We will now state some propositions regarding the simultaneous location decision problem corresponding to case 1 (the one formulated as S1). We will begin by formulating two auxiliary location bi-level decision problems that are directly related with problem S1. In problem S1 we are considering that both decision makers will decide simultaneously. In each of these two auxiliary problems[2], we will consider that these decisions are sequential: in one of the problems, the upper level considers the decisions made by player one and in the other problem the upper level considers the decisions made by the second player.

We will be in presence of a Nash equilibrium with pure strategies if there is a solution (y , w) that is the optimal solution to both of these problems.

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[2] For the sake of simplicity we consider total assignment in these formulations, but partial assignment could also be considered by replacing the assignment restrictions as shown before.

Aux-1

$$Max \sum_{i \in F \cup G} \sum_{j} (1-\alpha_1) d_{ij} x_{ij} \tag{2-3}$$

Subject to:

$$\sum_{i \in G} f_{i1} y_i \leq O_1 \tag{2-4}$$

$$y_i = 1, \forall i \in F \tag{2-5}$$

$$Max \sum_{i \in G} \sum_{j \in J} (1-\alpha_2) d_{ij} m_{ij} \tag{2-6}$$

Subject to:

$$y_i + z_i \leq 1, \forall i \in F \cup G \tag{2-7}$$

$$\sum_{i \in G} f_{i2} z_i \leq O_2 \tag{2-8}$$

$$m_{ij} \leq z_i, \forall i \in F \cup G, j \in J \tag{2-10}$$

$$x_{ij} \leq y_i, \forall i \in F \cup G, j \in J \tag{2-11}$$

$$\sum_{i \in F \cup G} (x_{ij} + m_{ij}) = 1, \forall j \in J \tag{2-12}$$

$$m_{ij} + x_{ij} \leq 1 - z_k - y_k, \forall i \in F \cup G, j \in J, k \in T_{ij} \tag{2-13}$$

$$z_i \leq w_i, \forall i \in F \cup G \tag{2-16}$$

$$z_i \geq w_i - y_i, \forall i \in F \cup G \tag{2-17}$$

$$z_i, y_i \in \{0,1\}, \forall i \in G$$

$$x_{ij} \in \{0,1\}, \forall i \in F \cup G, \forall j \in J$$

$$m_{ij} \in \{0,1\}, \forall i \in F \cup G, \forall j \in J$$

Aux-2

$$Max \sum_{i \in G} \sum_{j \in J} (1-\alpha_2) d_{ij} m_{ij} \tag{2-6}$$

Subject to:

$$\sum_{i \in G} f_{i2} w_i \leq O_2 \tag{2-14}$$

$$Max \sum_{i \in F \cup G} \sum_{j} (1-\alpha_1) d_{ij} x_{ij} \tag{2-3}$$

Subject to:

$$\sum_{i \in G} f_{i1} y_i \leq O_1 \tag{2-4}$$

$$y_i = 1, \forall i \in F \tag{2-5}$$

$$y_i + z_i \leq 1, \forall i \in F \cup G \tag{2-7}$$

$$m_{ij} \leq z_i, \forall i \in F \cup G, j \in J \tag{2-10}$$

$$x_{ij} \leq y_i, \forall i \in F \cup G, j \in J \tag{2-11}$$

$$\sum_{i \in F \cup G} (x_{ij} + m_{ij}) = 1, \forall j \in J \tag{2-12}$$

$$m_{ij} + x_{ij} \leq 1 - z_k - y_k, \forall i \in F \cup G, j \in J, k \in T_{ij} \tag{2-13}$$

$$z_i \leq w_i, \forall i \in F \cup G \tag{2-16}$$

$$z_i \geq w_i - y_i, \forall i \in F \cup G \tag{2-17}$$

$$z_i, y_i \in \{0,1\}, \forall i \in G$$

$$x_{ij} \in \{0,1\}, \forall i \in F \cup G, \forall j \in J$$

$$m_{ij} \in \{0,1\}, \forall i \in F \cup G, \forall j \in J$$

Considering a solution (y , w), let us also define the two following sets:

$G_1 = \{ i \in G : y_i = 1 \}$

$G_2 = \{ i \in G : w_i = 1 \}$

Given any (G_1, G_2), it is always possible to build the corresponding solution (y, w).

Let $\Omega_1(\mathrm{y}, \mathrm{w})$ represent the upper level objective function value of Aux-1, obtained with solution (y, w) and $\Omega_2(\mathrm{y}, \mathrm{w})$ represent the upper level objective function of Aux-2.

Proposition 1: Let (y , w) be an admissible solution to Aux-1, defined by (G_1, G_2). If the solution $(\mathrm{y}', \mathrm{w}')$ that corresponds to $(G_1 \cup \{i'\}, G_2), i' \notin G_1$, is also admissible, then $\Omega_1(\mathrm{y}', \mathrm{w}') \geq \Omega_1(\mathrm{y}, \mathrm{w})$.

Proof:

We should begin by noticing that if there is i such that $y_i = 1$ and $w_i = 1$, then $z_i = 0$. This means that $\Omega_1(\mathrm{y}, \mathrm{w})$ can be written as $\sum_{i \in F \cup G_1} \sum_j d_{ij} x_{ij}$.

Let $I(j) = \{ i \in F \cup G_1 \cup G_2 : d_{ij} \leq d_{kj}, \forall k \neq i, k \in F \cup G_1 \cup G_2 \}$. $I(j)$ represents the set of open facilities that are closest to *j*. Let $|I(j)|$ represent the number of elements in this set.

Let us assume that the assignment of clients to open facilities is done in such a way that if $|I(j)| > 1$ the demand of client *j* is equally distributed by all $i \in I(j)$, so:

$$x_{ij} = \begin{cases} \dfrac{1}{|I(j)|}, & \text{if } i \in I(j) \\ 0, & \text{otherwise} \end{cases}.$$

If we now consider the solution $(\mathrm{y}', \mathrm{w}')$ such that $y_{i'} = 1$, let us describe the changes that will occur in the corresponding assignment variables:

- There will possibly be a set of clients *j* such that $d_{i'j} > d_{kj}$, where *k* represents an open facility such that $x_{kj} > 0$. Let this set of clients be represented by J^+. If $j \in J^+$ then x_{ij} will not change.
- There will possibly be a set of clients *j* such that $d_{i'j} = d_{kj}$, where *k* represents an open facility such that $x_{kj} > 0$. Let this set of clients be represented by $J^=$, and additionally define $I_1(j) = \{ i \in I(j) : i \in G_1 \}$, $I_2(j) = \{ i \in I(j) : i \in G_2 \}$ and $I'(j) = \{ i \in F \cup G_1 \cup \{i'\} \cup G_2 : d_{ij} \leq d_{kj}, \forall k \neq i, k \in F \cup G_1 \cup \{i'\} \cup G_2 \}$. Then:

$$\sum_{i\in F\cup G_1\cup\{i'\}}\sum_{j\in J^=} d_{ij}x_{ij} = \sum_{j\in J^=}\sum_{i\in F\cup I_1(j)\cup\{i\}} d_{ij}\frac{1}{|I_1(j)|+1+|I_2(j)|} =$$
$$= \sum_{j\in J^=}\left(\sum_{i\in F\cup I_1(j)} d_{ij}\frac{1}{|I_1(j)|+1+|I_2(j)|} + d_{i'j}\frac{1}{|I_1(j)|+1+|I_2(j)|}\right) =$$
$$= \sum_{j\in J^=}\left(\sum_{i\in F\cup I_1(j)} d_{ij}\frac{1}{|I_1(j)|+1+|I_2(j)|} + d_{ij}\frac{1}{|I_1(j)|+1+|I_2(j)|}\right) = \sum_{j\in J^=} d_{ij}\frac{|I_1(j)|+1}{|I_1(j)|+1+|I_2(j)|}$$

Noticing that $\frac{|I_1(j)|+1}{|I_1(j)|+1+|I_2(j)|} \geq \frac{|I_1(j)|}{|I_1(j)|+|I_2(j)|}$, the changes in the assignment variables will never decrease the value of $\Omega_1(\mathrm{y},\mathrm{w})$.

- There will possibly be a set of clients *j* such that $d_{i'j} < d_{kj}$, where *k* represents an open facility such that either $x_{kj} > 0$ or $m_{kj} > 0$. Let this set of clients be represented by J^-. For each $j \in J^-$, if all $x_{kj} = 0$ for all $k \in F \cup G$ then, in solution $(\mathrm{y}', \mathrm{w}')$ player 1 will serve a client that he was not serving in solution (y, w), therefore increasing his demand. If we had $x_{kj} > 0$ for some $k \in F \cup G$, then player 1 will keep serving a client *j* that he was also serving before, but now he is doing it from a facility closer to *j*; therefore, according to (2-1) the demand from *j* to player 1 will not decrease .

As $J = J^+ \cup J^= \cup J^-$, we can conclude that $\Omega_1(\mathrm{y}', \mathrm{w}') \geq \Omega_1(\mathrm{y}, \mathrm{w})$. Notice that if $\Omega_1(\mathrm{y}', \mathrm{w}') = \Omega_1(\mathrm{y}, \mathrm{w})$, then the new facility i' will not capture any new demand, therefore it will not change player 2's payoff. In this case, any best response of player 2 to y is also a best response to y'.

A similar proposition can be stated regarding Aux-2. This means that there will be an optimal solution to problem Aux-1 and problem Aux-2 such that if we include one more location in sets G_1 or G_2 we will be violating the budget constraints.

Observation 1: If there is Nash equilibria with pure strategies, then there is at least one Nash equilibrium (y, w) such that changing a variable y_i from zero to one will violate the corresponding budget constraint and changing from 0 to 1 a variable w_i for which $y_i = 0$ will violate the corresponding budget constraint.

Proposition 2: Let (y , w) be an admissible solution to S1, based on sets (G_1, G_2). There will be w that is the best response to y such that $G_2 \subseteq \{i : i \notin G_1\}$.

Proof:

Facilities *i* such that $i \in G_2 \cap G_1$ will correspond to decision variables such that $w_i = 1$ but $z_i = 0$, and will not contribute to an improvement of $\Omega_2(\mathrm{y}, \mathrm{w})$.

To conclude this section, let us now describe two special cases, and prove the properties of these cases.

Special case 1: If player 1's budget constraint (2-4) is such that player 1 can open all facilities $(G_1 = G)$ then all solutions ($\mathbf{y}$, $\mathbf{w}$) such that $y_i = 1, \forall i \in G$ are Nash equilibrium solutions.

Proof: Notice that, in this case, it is obvious that the strategy of opening all facilities is optimal for player 1. When player 1 uses this strategy, the strategy of player 2 is irrelevant, since he will not be able to open any facility. Therefore, all solutions ($\mathbf{y}$, $\mathbf{w}$) such that $y_i = 1, \forall i \in G$ are Nash equilibrium solutions.

Special case 2: If, for every possible strategy $\mathbf{w}$ of player 2, the best response of player 1 is always $\mathbf{y}$, and just $\mathbf{y}$, then any optimal solution of either Aux-1 or Aux-2 is a Nash equilibrium solution to both S1 and S2.

Proof: In an optimal solution of Aux-1, player 1 will choose $\mathbf{y}$, which is the best response to any strategy of player 2, and then player 2 will choose the best response to $\mathbf{y}$. Therefore we get a Nash equilibrium.

In an optimal solution to Aux-2, player 2 plays first but he knows that player 1 will surely choose $\mathbf{y}$, so he will choose the best response to $\mathbf{y}$. Player 1 will then choose $\mathbf{y}$, which is the best response to any strategy of player 2. Therefore we get a Nash equilibrium.

4 Concluding remarks

In this research report we have described a simultaneous decision location problem, in a competitive setting, where two decision makers have to decide simultaneously where to locate facilities, but at most one facility can be opened in each potential location. This means that if both decision makers are interested in the same location, only one of them will be able to open the facility.

We have presented a possible linear programming formulation for this problem, and explored the connections between this problem and other related bi-level decision problems.

In this research report we have only considered the existence of Nash equilibrium with pure strategies. Nevertheless, we know that a Nash equilibrium is guaranteed to exist, possibly with mixed strategies. Following the work of Godinho and Dias, 2010, an algorithm can be developed to find these equilibria.

It will be of particular interest to analyse what happens if we consider the location of essential services such that demand will not be a function of the distance.

References

Bard, J. F. and J. T. Moore (1992). "An Algorithm for the Discrete Bilevel Programming Problem." Naval Research Logistics **39**: 419-435.

Godinho, P. and J. Dias (2010). "A two-player competitive discrete location model with simultaneous decisions." European Journal of Operational Research **207**:1419-1432

Eiselt, H. A. (1998). "Perception and information in a competitive location model." European Journal of Operational Research **108**(1): 94-105.

Eiselt, H. A. and G. Laporte (1996). "Sequential Location Problems." European Journal of Operational Research **96**: 217-231.

Eiselt, H. A., G. Laporte, et al. (1993). "Competitive Location Models: A Framework and Bibliography." Transportation Science **27**(1): 44-54.

Hotelling, H. (1929). "Stability in competition." The economic journal: 41-57.

Plastria, F. (2001). "Static competitive facility location: An overview of optimization approaches." European Journal of Operational Research **129**: 461-470.