Capacitated Dynamic Location Problems with Opening, Closure and Reopening of Facilities

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Abstract: In this research report three capacitated dynamic location problems with opening, closure and reopening of facilities are formulated and primal-dual heuristics are described that can solve those problems. In the first problem addressed, maximum capacity restrictions are considered and, in the second problem, both maximum and minimum capacity restrictions are considered. The last problem formulated considers the situation where a facility is open (or reopen) with a certain maximum capacity that decreases as clients are assigned to that facility during its operating periods. All problems formulated are \(NP\)-hard. Primal-dual heuristics were developed that can calculate admissible solutions for these location problems.

1 Introduction

In the simple capacitated location problem a set of clients with known demands has to be assigned to a set of facilities with maximum capacities. The objective function considers assignment costs and facilities’ fixed opening costs. Generally it is considered that a client can be assigned to one or more facilities (which is called partial assignment). In this case, after fixing as open a subset of facilities, the optimal assignment of clients to open facilities corresponds to the optimal solution of a transportation problem. A harder version of the

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capacitated location problem consists in considering that a client has to be served by exactly one facility (total assignment).

There are several references in the literature that deal with capacitated location problems. Cornuejols et al (1991) perform a systematic study of the relative quality of heuristics and relaxations described in the literature. Sridharan (1995) provides a review of both heuristic and exact methods that deal with capacitated location problems.

Guignard and Spielberg (1979) describe an algorithm based on the dual problem of the linear relaxation of a capacitated facility location problem. The method described permits the calculation of tight lower bounds, and the construction of feasible solutions based on the dual solution found. The authors’ formulation of the problem includes an additional restriction that can be used to impose limits on the number of open or closed plants, or to impose any a priori side constraints on the facilities. Bitran et al (1981) use the capacitated plant location problem to illustrate the inverse optimisation technique. Jacobsen (1983) generalizes several well-known heuristics used in the uncapacitated case to the capacitated location problem. Christofides and Beasley (1983) use lagrangean relaxation and subgradient optimisation to find lower bounds for the capacitated location problem. The lower bounds calculated are then used in a branch and bound procedure that allows the calculation of the optimal solution for problems with 50 facility locations and 150 clients. Van Roy (1986) describes the cross decomposition method to solve the capacitated location problem. This method uses both Benders Decomposition and Lagrangean relaxation. Computational results are shown for problems with up to 100 facility potential locations and 200 clients. Baker (1986) describes a branch and bound procedure that uses a partial dual of a tight LP formulation, allowing the use of efficient algorithms for transportation problems. Beasley (1988) describes a tree search procedure that is able to solve problems with 500 potential facilities and 1000 clients. This procedure is based on a Lagrangean relaxation. Barceló and Casanovas (1984), Klincewicz and Luss (1986) and also Pirkul (1987) study the capacitated location problem in which a client can only be served by one facility. The authors use Lagrangean relaxation methods to deal with the problem. Barceló et al (1991) use the “variable splitting” technique and lagrangean relaxation, dealing with both mixed and pure integer capacitated location problems. A branch and bound method based on a Lagrangean heuristic that guarantees the calculation of the optimal solution to the capacitated location problem with single sourcing can be found in Holmberg et al (1999). Cortinhal and Captivo (2003 a, b) study the total assignment capacitated location problem, using lagrangean relaxation, tabu search and genetic algorithms. Baldacci et al (2002) consider
the capacitated $p$-median problem, with total assignment, and present an exact algorithm based on a set partitioning formulation of the problem.

Lee (1991) describes a generalization of capacitated facility location problem considering that facilities can provide several different products, having a maximum capacity established by product. Clients have to be assigned to particular products at given facilities. The author considers an added fixed cost incurred for every product a facility can handle. The problem is solved by Benders’s decomposition. Mazzola and Neebe (1999) studied the same problem, describing a lagrangean-based heuristic and also a branch and bound procedure. Bloemhof-Ruwaard et al (1996) describe a problem that is a generalization of the two-level capacitated facility location problem: the capacitated distribution and waste disposal problem. Pirkul and Jayaraman (1998) consider the multi-commodity version of the capacitated plant and warehouse location problem (customers are served from open warehouses which, in turn, receive products from manufacturing plants), and develop an efficient heuristic based on lagrangean relaxation. Hinojosa et al (2000) use a lagrangean relaxation based heuristic to find feasible solutions to a two-level, multicommodity capacitated problem. Agar and Salhi (1998) describe several lagrangean heuristics that can tackle large capacitated plant location problems (up to 100 facilities and 1000 clients). The authors describe computational experiments considering three different problems: the capacitated location problem with partial assignment, with total assignment and also the multi-capacitated plant location problems. Melkote and Daskin (2001) treat the capacitated version of the facility location/network design problems (that includes the classical capacitated facility location problem), presenting several classes of valid inequalities that strengthen problem’s LP relaxation. Ghiani et al (2002) treat the problem of capacitated plant location problem with multiple facilities in the same site as a generalization of the capacitated facility location problem.

There are quite fewer references to the dynamic capacitated location problem than to the static version of the problem. Erlenkotter (1975) uses dynamic programming to handle the problem of capacity planning for large multilocation systems: these are dynamic capacity planning problems with many locations. Fong and Srinivasan (1981a,b) treat the problem of capacity expansion over a planning horizon. Luss (1982) treats the problem of planning capacity expansion of facilities over a time horizon. Van Roy and Erlenkotter (1982) give some ideas of how to use a dual ascent method to solve the dynamic capacitated facility location problem. Min (1988) describes the problem of dynamic expansion and relocation of capacitated public facilities. Shulman (1991) studies the problem of locating capacitated facilities during a planning horizon, but allowing several facilities to be located at the same site, in different time
periods. This problem appears whenever it is necessary to consider the discrete expansion of the maximum capacity at one particular location. Saldanha da Gama (2002) and Saldanha da Gama and Captivo (2002) describe a branch and bound procedure for the dynamic capacitated location problem. The problem studied considers that a facility that is open at the beginning of the planning horizon can be closed (remaining closed until the end of the planning horizon), and facilities that are open at the beginning of a time period $t$ remain open until the end of the planning horizon. The capacity of a facility cannot be changed during its operating lifetime.

The problems studied in this research report have one important characteristic that distinguish them from the previous work done in this area: they are capacitated dynamic location problems that consider the possibility of reconfiguring one location more than once during the planning horizon. This means that a facility can be open, closed and reopen more than once, which increases the flexibility of the models. Differentiation between the opening and the reopening of a facility is convenient because it allows the differentiation of the corresponding fixed costs (that can be clearly different). The models proposed also consider the existence of closing costs which, most of the times, cannot be ignored.

The primal-dual heuristics developed are based on the work of Erlenkotter (1978) and Guignard and Spielberg (1979). They build a pair of primal and dual solutions, trying to force the complementary conditions to be fulfilled.

In the next three sections, the three problems addressed are formulated and the corresponding linear dual problems are presented. In section 5, the primal-dual heuristics are described. In section 6 some final comments are made and future work directions are pointed out.

2 Dynamic Location Problem with Maximum Capacity Constraints

Consider the following notation:

- $J = \{1,...,n\}$ set of indexes corresponding to the clients’ locations;
- $I = \{1,...,m\}$ set of indexes corresponding to facilities’ possible locations;
- $T$ = number of time periods considered in the planning horizon;
- $c_{ij}^t$ = cost of fully assigning client $j$ to facility $i$ in period $t$;
- $FA_{it}^\xi$ = fixed cost of opening a facility $i$ at the beginning of period $t$, and closing it at the end of period $\xi$ (the facility will be in operation from the beginning of $t$ to the end of $\xi$);
\( FR_i^x = \) fixed cost of reopening a facility \( i \) at the beginning of period \( t \), and closing it at the end of period \( \xi \) (the facility will be in operation from the beginning of \( t \) to the end of \( \xi \)); 
\( d_j^t = \) demand of client \( j \) at period \( t \); 
\( Q_i = \) maximum capacity of facility located at \( i \). 
and let us define the variables:

\[
a_{it}^\xi = \begin{cases} 
1 & \text{if facility } i \text{ is open at the beginning of period } t \text{ and stays open until the end of period } \xi \\
0 & \text{otherwise}
\end{cases}
\]

\[
r_{it}^\xi = \begin{cases} 
1 & \text{if facility } i \text{ is reopen at the beginning of period } t \text{ and stays open until the end of period } \xi \\
0 & \text{otherwise}
\end{cases}
\]

\( x_{ij}^t = \) fraction of customer \( j \)'s demand that is served by facility \( i \) during period \( t \).

The dynamic capacitated location problem that allows facilities to open, close and reopen more than once during the planning horizon will be formulated as C1-DLPOCR:

**C1-DLPOCR**

\[
\begin{align*}
\text{Min} & \sum_t \sum_i \sum_j c_{ij}^t x_{ij}^t + \sum_t \sum_i \sum_{\xi=t}^T FA_i^\xi a_{it}^\xi + \sum_t \sum_i \sum_{\xi=t}^T FR_i^\xi r_{it}^\xi \\
\text{subject to:} & \\
\sum_i x_{ij}^t &= 1, \quad \forall j, t \quad (2) \\
\sum_{\tau=1}^t \sum_{\xi=t}^T \left( a_{it}^\xi + r_{it}^\xi \right) x_{ij}^t &\geq 0, \quad \forall i, j, t \quad (3)
\end{align*}
\]

\(^1\) In this model it is considered that the maximum capacity of a facility located at \( i \) will remain constant during the planning horizon. This means that the maximum capacity of a given facility is the same during all its operating periods. It is also possible to consider that this maximum capacity can change over the planning horizon. In this case capacities \( Q_i^t \) should be considered. This change can easily be incorporated in the procedures that are going to be presented.
Constraints (2) guarantee that, in every time period, each client’s demand is satisfied; constraints (3) assure that, in every time period, a client can only be assigned to facilities that are operational in that time period; constraints (4) impose that a facility can only be reopen at the beginning of period $t$ if it has already been open earlier and is not in operation at the beginning of period $t$; constraints (5) guarantee that a facility can only be open once during the planning horizon; constraints (6) assure that, in every time period, only one facility can be open in each location. Constraints (7) impose that the maximum capacity of a facility that is operational during time period $t$ will not be exceeded.

**Formulation of the Dual Problem**

Let us rewrite constraints (5) and (6) equivalently as (5’) and (6’):

$$
\sum_{t=1}^{T} \sum_{i,t} a_{it}^{\xi} \leq 1, \quad \forall i
$$

$$
\sum_{t=1}^{T} \sum_{i,t} \left( a_{it}^{\xi} + r_{it}^{\xi} \right) \leq 1, \quad \forall i, t
$$

Considering dual variables $v_j^t$ associated with constraints (2), dual variables $w_{ij}^t$ associated with constraints (3), dual variables $u_i^t$ associated with constraints (4), dual variables $\rho_i$ associated with constraints (5’), dual variables $\pi_i^t$ associated with constraints (6’), and dual
variables $\lambda_i^t$ associated with constraints (7), the dual problem of C1-DLPOCR can be formulated as DC1-DLPOCR:

**DC1-DLPOCR**

$$\text{Max} \sum_t \sum_j v_j^t - \sum_i \rho_i - \sum_t \sum_i \lambda_i^t$$  \hspace{1cm} (9)

subject to:

$$v_j^t - w_j^t - d_j^t \lambda_i^t \leq c_j^t, \hspace{1cm} \forall i, j, t$$  \hspace{1cm} (10)

$$\sum_t \sum_{j \tau=t}^T w_{ij}^\tau + \sum_{\tau=t}^{\tau+1} u_i^\tau - \rho_i - \sum_{\tau=t}^\tau \sum_i \pi_i^\tau + Q_i \sum_{\tau=t}^\xi \lambda_i^\tau \leq FA_i^\xi, \hspace{1cm} \forall i, t, \xi = t, \ldots, T$$  \hspace{1cm} (11)

$$\sum_t \sum_{j \tau=t}^T w_{ij}^\tau - u_i^t - \sum_{\tau=t}^\tau \sum_i \pi_i^\tau + Q_i \sum_{\tau=t}^\xi \lambda_i^\tau \leq FR_i^\xi, \hspace{1cm} \forall i, t > 1, \xi = t, \ldots, T$$  \hspace{1cm} (12)

$$w_{ij}^t, u_i^t, \rho_i, \pi_i^t, \lambda_i^t \geq 0, \hspace{1cm} \forall i, j, t$$

Considering $w_{ij}^t = \max\{0, v_j^t - c_j^t - d_j^t \lambda_i^t\}$, an equivalent condensed formulation is obtained:

**CDC1-DLPOCR**

$$\text{Max} \sum_t \sum_j v_j^t - \sum_i \rho_i - \sum_t \sum_i \pi_i^t$$

subject to:

$$\sum_t \sum_{j \tau=t}^\xi \max\{0, v_j^\tau - c_j^t - d_j^t \lambda_i^t\} \leq FA_i^\xi - \sum_{\tau=t}^{\tau+1} u_i^\tau + \sum_{\tau=t}^\tau \sum_i \pi_i^\tau - Q_i \sum_{\tau=t}^\xi \lambda_i^\tau, \hspace{1cm} \forall i, t, \xi = t, \ldots, T$$  \hspace{1cm} (13)

$$\sum_t \sum_{j \tau=t}^T \max\{0, v_j^\tau - c_j^t - d_j^t \lambda_i^t\} \leq FR_i^\xi + u_i^t + \sum_{\tau=t}^\tau \sum_i \pi_i^\tau - Q_i \sum_{\tau=t}^\xi \lambda_i^\tau, \hspace{1cm} \forall i, t > 1, \xi = t, \ldots, T$$  \hspace{1cm} (14)

$$u_i^t, \rho_i, \pi_i^t, \lambda_i^t \geq 0, \hspace{1cm} \forall i, j, t$$
Complementary Conditions

Let us define:

\[
SA^\xi_{it} = FA^\xi_{it} - \sum_{\tau=\xi+1}^T u^\tau_i + \rho_i + \sum_{\tau=t}^\xi \pi^\tau_i - \sum_{\tau=t}^\xi \sum_{j=t}^\xi \max\left\{0, v^\tau_j - c^\tau_{ij} - d^\tau_{ij} \lambda^\tau_j\right\} - Q_i \sum_{\tau=t}^\xi \lambda^\tau_i ,
\]

\forall i, t, \xi = t, \cdots, T \quad (15)

\[
SR^\xi_{it} = FR^\xi_{it} + u^t_i + \sum_{\tau=t}^\xi \pi^\tau_i - \sum_{\tau=t}^\xi \sum_{j=t}^\xi \max\left\{0, v^\tau_j - c^\tau_{ij} - d^\tau_{ij} \lambda^\tau_j\right\} - Q_i \sum_{\tau=t}^\xi \lambda^\tau_i ,
\]

\forall i, t > 1, \xi = t, \cdots, T \quad (16)

\[
S^\xi_{it} = \min\{SA^\xi_{it}, SR^\xi_{it}\} ,
\]

\forall i, t, \xi = t, \cdots, T \quad (17)

Considering primal problem C1-DLPOCR and its dual problem CDC1-DLPOCR, the following complementary conditions hold if in presence of optimal primal and dual solutions to the respective problems (when there is no duality gap).

\[
\left(\sum_{\tau=1}^T \sum_{\xi=t}^\tau \left(a^\xi_{i\tau} + r^\xi_{i\tau}\right)\right)w_{ij} = 0 ,
\]

\forall i, j, t \quad (18)

\[
\left(\sum_{\tau=1}^{t-1} \sum_{\xi=t}^{\tau} \sum_{\xi=t}^{\tau} a^\xi_{i\tau} - \sum_{\xi=t}^{\tau} r^\xi_{i\tau} \right)u^t_i = 0 ,
\]

\forall i, t \quad (19)

\[
\left(\sum_{\tau=1}^T \sum_{\xi=t}^\tau a^\xi_{it} - 1 \right)\rho_i = 0 ,
\]

\forall i \quad (20)

\[
\left(\sum_{\tau=1}^T \sum_{\xi=t}^\tau \left(a^\xi_{i\tau} + r^\xi_{i\tau}\right) - 1 \right)\pi^t_i = 0 ,
\]

\forall i, t \quad (21)

\[
SA^\xi_{it} \cdot a^\xi_{it} = 0 ,
\]

\forall i, t, \xi = t, \cdots, T \quad (22)

\[
SR^\xi_{it} \cdot r^\xi_{it} = 0 ,
\]

\forall i, t > 1, \xi = t, \cdots, T \quad (23)

\[
Q_i \sum_{\tau=1}^T \sum_{\xi=t}^\tau \left(a^\xi_{i\tau} + r^\xi_{i\tau}\right) - \sum_{j} d^t_{ij} x^t_{ij} \lambda^t_i = 0 ,
\]

\forall i, t \quad (24)
3 Dynamic Location Problem with Maximum and Minimum Capacity Constraints

Considered the notation defined in the previous section and:

\( Q'_i \) = minimum functioning capacity of facility at location \( i \).

The dynamic capacitated problem that considers both maximum and minimum capacity restrictions can be formulated as:

\[
\text{C2-DLPOCR}
\]

\[
\min \sum_{t} \sum_{i} \sum_{j} c_{ij}^t x_{ij}^t + \sum_{t} \sum_{i} \sum_{\xi=t}^T F A_{it}^\xi a_{iit}^\xi + \sum_{t} \sum_{i} \sum_{\xi=t}^T F R_{it}^\xi r_{iit}^\xi
\]

subject to:
(2) – (8)

\[
\sum_{j} d_{ij}^t x_{ij}^t - Q'_i t \sum_{\tau=1}^{t} \left( a_{i\tau}^\tau + r_{i\tau}^\tau \right) \geq 0, \quad \forall i, t \quad (25)
\]

Restrictions (25) guarantee that, if facility located at \( i \) is in operation during time period \( t \), than it has to serve at least \( Q'_i \) units of clients’ demands. This kind of restrictions is important to be considered, because in almost all situations a facility has to operate above a minimum level of service, or else it is not economically viable.

Associating dual variables \( \beta_i^t \) with constraints (25) a condensed dual linear problem of C2-DLPOCR can be formulated as:

\[
\text{CDC2-DLPOCR}
\]

\[
\max \sum_{t} \sum_{j} v_{ij}^t - \sum_{i} \rho_i - \sum_{t} \sum_{i} \pi_{it}^t
\]

subject to:

---

\(^2\) The observation made in footnote 1 is also valid in this case. It is considered that \( Q'_i \leq Q_i, \forall i \). Otherwise the problem would be impossible.
\[
\sum_{j} \sum_{\tau=t}^{\xi} \max \{0, v_j^{\tau} - c_{ij}^{\tau} - d_j^{\tau} \lambda_{ij}^{\tau} + d_j^{\tau} \beta_{ij}^{\tau}\} \leq FA_{it}^{\xi} - \sum_{\tau=t+1}^{T} u_{i\tau} + \rho_i + \sum_{\tau=t}^{T} \pi_{i\tau} - Q_i \sum_{\tau=t}^{\xi} \lambda_{i\tau}^{\tau} + Q_i^{'} \sum_{\tau=t}^{\xi} \beta_{i\tau}^{\tau}, \\
\forall i, t, \xi = t, \cdots, T
\]
\[
\sum_{j} \sum_{\tau=t}^{\xi} \max \{0, v_j^{\tau} - c_{ij}^{\tau} - d_j^{\tau} \lambda_{ij}^{\tau} + d_j^{\tau} \beta_{ij}^{\tau}\} \leq FR_{it}^{\xi} + u_{i1}^{t} + \sum_{\tau=t}^{\xi} \pi_{i\tau} - Q_i \sum_{\tau=t}^{\xi} \lambda_{i\tau}^{\tau} + Q_i^{'} \sum_{\tau=t}^{\xi} \beta_{i\tau}^{\tau}, \\
\forall i, t > 1, \xi = t, \cdots, T
\]
\[
u_{it}, \rho_i, \pi_{i\tau}, \lambda_{i\tau}^{\tau}, \beta_{i\tau}^{\tau} \geq 0, \\
\forall i, j, t
\]

Defining:

\[
SA_{it}^{\xi} = FA_{it}^{\xi} - \sum_{\tau=t+1}^{T} u_{i\tau} + \rho_i + \sum_{\tau=t}^{T} \pi_{i\tau} - \sum_{j} \sum_{\tau=t}^{\xi} \max \{0, v_j^{\tau} - c_{ij}^{\tau} - d_j^{\tau} \lambda_{ij}^{\tau} + d_j^{\tau} \beta_{ij}^{\tau}\} - Q_i \sum_{\tau=t}^{\xi} \lambda_{i\tau}^{\tau} + Q_i^{'} \sum_{\tau=t}^{\xi} \beta_{i\tau}^{\tau}, \\
\forall i, t, \xi = t, \cdots, T
\]
\[
SR_{it}^{\xi} = FR_{it}^{\xi} + u_{i1}^{t} + \sum_{\tau=t}^{\xi} \pi_{i\tau} - \sum_{j} \sum_{\tau=t}^{\xi} \max \{0, v_j^{\tau} - c_{ij}^{\tau} - d_j^{\tau} \lambda_{ij}^{\tau} + d_j^{\tau} \beta_{ij}^{\tau}\} - Q_i \sum_{\tau=t}^{\xi} \lambda_{i\tau}^{\tau} + Q_i^{'} \sum_{\tau=t}^{\xi} \beta_{i\tau}^{\tau}, \\
\forall i, t > 1, \xi = t, \cdots, T
\]

Complementary conditions (18) – (24) hold, and there is one more to be considered:

\[
\left( \sum_{j} d_j^{t} x_{ij}^{t} - Q_i \sum_{\tau=1}^{T} \left( a_{i\tau}^{\xi} + r_{i\tau}^{\xi} \right) \right) \beta_{i\tau}^{\xi} = 0, \\
\forall i, t
\]

4 Dynamic Location Problem with Maximum Decreasing Capacities

Consider the notation defined in section 2, reinterpreting \( Q_i \) as the maximum capacity of facility \( i \) at the opening time period or the maximum capacity expansion of facility \( i \) at the time of reopening. Consider the following problem:

C3-DLPOCR

\[
\text{Min} \sum_{t} \sum_{i} \sum_{j} c_{ij}^{t} x_{ij}^{t} + \sum_{t} \sum_{i} \sum_{\xi=t}^{T} FA_{it}^{\xi} a_{i\tau}^{\xi} + \sum_{t} \sum_{i} \sum_{\xi=t}^{T} FR_{it}^{\xi} r_{i\tau}^{\xi}
\]
subject to:

(2) – (6), (8)

\[ Q_i \sum_{\tau=1}^{T} \left( a_{i\tau}^\xi + r_{i\tau}^\xi \right) - \sum_{\tau=1}^{T} \sum_{j} d_{j}^\tau x_{ij}^\tau \geq 0, \quad \forall i, t \quad (31) \]

This problem describes the situation where a service can be open (or reopen) with a certain maximum capacity. As long as this facility serves clients’ demands, its capacity decreases. Examples of facilities with this kind of behavior can be found, for instance, in sanitary landfills. When these facilities are open, they can receive a maximum quantity of solid waste. This maximum capacity decreases during the life-period of the sanitary landfill, as it receives solid waste.

The model presented considers admissible the situation where a facility is closed even if its capacity has not been totally used. Restriction (31) considers that when a facility is reopened, its maximum capacity will be equal to \( Q_i \) plus the remaining capacity the facility had when it was closed. It can be argued that this behavior is not admissible for some kinds of facilities. Thinking, for instance, of sanitary landfills it is easy to imagine that if a sanitary landfill is closed at period \( t \) and reopened at period \( t+1 \), then its remaining capacity at the end of \( t \) can be used. Nevertheless, if the sanitary landfill is reopen several time periods after its closure, its remaining capacity at the end of period \( t \) will have been lost (because of all the closing and maintenance operations that need to be performed). The fixed opening and reopening costs of these kind of facilities are generally huge when compared with transportation and handling costs, so it is not expected that a facility with useful remaining capacity will be closed, unless the remaining capacity is insignificant when compared with \( Q_i \). Furthermore, the decision maker is free to consider only variables \( a_{i\tau}^\xi \) and \( r_{i\tau}^\xi \) that he feels are needed. He can, for instance, consider variables such that \( \xi - \tau \) is greater than a minimum time interval. For these reasons, the authors feel that the model presented has an acceptable behavior and can be considered as useful in the resolution of many real problems.

Associating dual variables \( \lambda_i^\tau \) with restrictions (31), the dual linear condensed problem can be formulated as:

\[ Q_i \sum_{\tau=1}^{T} \left( a_{i\tau}^\xi + r_{i\tau}^\xi \right) - \sum_{\tau=1}^{T} \sum_{j} d_{j}^\tau x_{ij}^\tau \geq 0, \quad \forall i, t \quad (31) \]
CDC3-DLPOCR

Max $\sum_t \sum_j v_j^t - \sum_i \rho_i - \sum_i \pi_i^t$

subject to:

$\sum_j \sum_{\tau=t}^T \max \left\{ 0, v_j^\tau - c_{ij} - \sum_{\psi=\tau}^T d_j^\psi \lambda_i^\tau \right\} \leq FA_{it}^\xi - \sum_{\tau=\xi+1}^T u_i^\tau + \sum_i \pi_i^t - Q_i \sum_{\tau=t}^T \lambda_i^t$

$\forall i, t, \xi = t, \ldots, T$ (32)

$\sum_j \sum_{\tau=t}^T \max \left\{ 0, v_j^\tau - c_{ij} - \sum_{\psi=\tau}^T d_j^\psi \lambda_i^\tau \right\} \leq FR_{it}^\xi + u_i^t + \sum_i \pi_i^t - Q_i \sum_{\tau=t}^T \lambda_i^t$

$\forall i, t > 1, \xi = t, \ldots, T$ (33)

$u_i^t, \xi, \pi_i^t, \lambda_i^t \geq 0$

Defining:

$SA_{it}^\xi = FA_{it}^\xi - \sum_{\tau=\xi+1}^T u_i^\tau + \sum_i \pi_i^t - \sum_j \sum_{\tau=t}^T \max \left\{ 0, v_j^\tau - c_{ij} - \sum_{\psi=\tau}^T d_j^\psi \lambda_i^\tau \right\} - Q_i \sum_{\tau=t}^T \lambda_i^t$

$\forall i, t, \xi = t, \ldots, T$ (34)

$SR_{it}^\xi = FR_{it}^\xi + u_i^t + \sum_i \pi_i^t - \sum_j \sum_{\tau=t}^T \max \left\{ 0, v_j^\tau - c_{ij} - \sum_{\psi=\tau}^T d_j^\psi \lambda_i^\tau \right\} - Q_i \sum_{\tau=t}^T \lambda_i^t$

$\forall i, t > 1, \xi = t, \ldots, T$ (35)

In presence of a pair of primal and dual admissible solutions, complementary conditions (18) – (23) have to be satisfied, plus the following condition:

$\left( \sum_{\tau=1}^T \sum_{\xi=1}^T (Q_i \sum_{\tau=1}^T (a_{it}^\xi + r_{it}^\xi) - \sum_{\tau=1}^T \sum_j d_j^\xi x_{ij}^\xi) \lambda_i^t \right) = 0$

$\forall i, t$ (36)
5 Primal-Dual Heuristics

The primal-dual heuristics that were developed to solve the three problems formulated in the previous sections build admissible primal solutions based on admissible dual solutions, trying to force the complementary conditions to be satisfied. The heuristics are very similar to each other, and they only differ in some of the dual procedures and in the primal procedure. The heuristics functioning scheme is the following:

1. Initialisation of dual variables;
2. Dual Ascent Procedure for dual variables $v_j^i$;
3. Primal Procedure;
4. Dual Adjustment Procedure for dual variables $\rho_i$. If the dual solution is changed go to 2;
5. Repeat the Dual-Primal Adjustment Procedure for variables $v_j^i$ until there is no improvement in the dual objective function value;
6. Dual Adjustment Procedure for dual variables $\rho_i$. If the dual solution is changed go to 2;
7. Dual Ascent Procedure for dual variables $u_t^i$. If the dual solution is changed go to 2;
8. Dual Descent Procedure for dual variables $u_t^i$. If the dual solution is changed go to 2;
9. Dual Ascent Procedure for dual variables $\lambda_t^i$. If the dual solution is changed go to 2;
10. Dual Descent Procedure for dual variables $\lambda_t^i$. If the dual solution is changed go to 2;
11. Dual Ascent Procedure for dual variables $\beta_t^i$. If the dual solution is changed go to 2;\textsuperscript{3}
12. Dual Descent Procedure for dual variables $\beta_t^i$. If the dual solution is changed go to 2;\textsuperscript{3}
13. Dual Adjustment Procedure for variables $\pi_t^i$. If the dual solution is changed go to 2.

The heuristic will stop when the optimal solution is found (the pair of primal and dual solutions satisfies all complementary conditions), or when there are no improvements in either primal or dual objective function values.

Steps 1, 4-8 and 13 are exactly the same as described in Dias et al (2002) for the dynamic location problem with opening, closure and reopening of facilities, without capacity restrictions (DLPOCR). Step 2 of the primal–dual heuristics is the same described in Dias et al (2002),

\textsuperscript{3} Steps 11 and 12 are only executed for C2-DLPOCR.
considering the assignment costs for period $t$ as
\[ c_{ij}^t + d_j^t \lambda_i^t \] for C1-DLPOCR, $c_{ij}^t + d_j^t \left( \lambda_i^t - \beta_i^t \right)$ for C2-DLPOCR and $c_{ij}^t + d_j^t \sum_{\tau=t}^{T} \lambda_{\tau}^t$ for C3-DLPOCR.

In the following sections, steps 3 and 9-12 will be described for each of the three problems.

**Maximum capacity restrictions**

Expressions (15) and (16) show the relation of slacks $\xi_{SA_{t}\tau}$ and $\xi_{SR_{t}\tau}$ with the dual variables $\lambda_i^t$. A change in the dual variable $\lambda_i^t$ will influence all slacks $SA_{t}\tau$ and $SR_{t}\tau$, with $\tau \leq t \leq \xi$. This dual variable contributes to each slack with the value:

$$\sum_j \max\left\{ 0, v_j^t - c_{ij}^t - d_j^t \lambda_i^t \right\} + Q_i \lambda_i^t$$  \hspace{1cm} (37)

Therefore, slacks can be increased with both the increase and the decrease of the dual variable value. The increase of $\lambda_i^t$ will increase the second part of expression (37) and decrease the first one. The decrease of $\lambda_i^t$ will have the opposite effect.

**Dual Ascent Procedure for variables $\lambda_i^t$**

Consider the following definitions:

$$J(\delta) = \left\{ j \in J : \frac{1}{d_j^t} \left( v_j^t - c_{ij}^t - d_j^t \lambda_i^t \right) \leq \delta \right\}$$  \hspace{1cm} (38)

$$\Lambda = \max_{j \in J} \left\{ \frac{1}{d_j^t} \max\left\{ 0, v_j^t - c_{ij}^t - d_j^t \lambda_i^t \right\} \right\}$$  \hspace{1cm} (39)

**Proposition 1:** Consider $\delta \in ]0,\Lambda]$, with $\Lambda$ defined as in (39). Let variable $\lambda_i^t$ be increased by $\delta$. If

$$Q_i \delta < \sum_{j \in J(\delta)} \max\left\{ 0, v_j^t - c_{ij}^t - d_j^t \lambda_i^t \right\} + \sum_{j \notin J(\delta)} d_j^t \delta$$

then all slacks $SA_{t}\tau$ and $SR_{t}\tau$ with $\tau \leq t \leq \xi$ will be increased by:

$$\Omega(\delta) = \sum_{j \in J(\delta)} \max\left\{ 0, v_j^t - c_{ij}^t - d_j^t \lambda_i^t \right\} + \sum_{j \notin J(\delta)} d_j^t \delta - Q_i \delta$$  \hspace{1cm} (40)
This result follows similar results that can be found in Guignard and Spielberg (1979) and Saldanha da Gama (2002). Therefore the proof is omitted.

**Proposition 2:** For every \( \delta' > \Delta \) it is always possible to find a \( \delta \leq \Delta \) so that \( \Omega(\delta) > \Omega(\delta') \). So the maximum value \( \delta \) should take is given by \( \Delta \).

The proof follows directly from proposition 1 and (39).

The dual ascent procedure for variables \( \lambda_i^t \) is based on Proposition 1 and follows the work of Saldanha da Gama (2002).

**Dual Ascent Procedure for Variables \( \lambda_i^t \)**

1. \( t \leftarrow 1; \)
2. \( i \leftarrow 1; \)
3. \( \delta \leftarrow 0; \delta' \leftarrow +\infty; \text{changed} \leftarrow \text{false}; \)

4. \( \delta \leftarrow \max_{j \in J} \left\{ \frac{1}{d_j^t} \max \left\{ 0, v_j^t - c_{ij}^t - d_j^t \lambda_j^t \right\}, \delta' \right\}; \) If \( \delta = 0 \), then go to 9. Compute \( J_1(\delta) \) as in (38).

5. If \( \Omega(\delta) > 0 \) then go to 7, else go to 8.

6. If \( \Omega(\delta) > 0 \) then go to 7, else go to 8.

7. \( SA_{t_a}^x \leftarrow SA_{t_a}^x + \Omega(\delta), \ SR_{t_a}^x \leftarrow SR_{t_a}^x + \Omega(\delta^t), \forall \tau, \xi \) such that \( \tau \leq t \leq \xi, \lambda_i^t \leftarrow \lambda_i^t + \delta; \) \text{changed} \leftarrow \text{true}; \) go to 9.

8. \( \delta' \leftarrow \delta; \)

9. if \( \delta \neq 0 \) and not \text{changed} then go to 3. Else \( i \leftarrow i + 1; \)

10. If \( i > M \), then \( t \leftarrow t + 1. \) Else go to 3.

11. If \( t > T \), then stop. Else go to 2.
Dual Descent Procedure for variables $\lambda^I_i$

All values $v_j^I - c_{ij}^I - d_j^I \lambda^I_i$ will be increased with the decrease in the dual variable $\lambda^I_i$.

For values $v_j^I - c_{ij}^I - d_j^I \lambda^I_i \geq 0$, this increase can be calculated as $\delta d_j^I$, if the change in the dual variable is equal to $\delta$.

If $\delta$ is such that all values $v_j^I - c_{ij}^I - d_j^I \lambda^I_i < 0$ remain less than zero, then the change in slacks $SA_{i\tau}^\xi$ and $SR_{i\tau}^\xi$, with $\tau \leq t \leq \xi$ is given by

$$\delta \left( Q_i - \sum_{j \in J} d_j^I \right)$$

$$v_j^I - c_{ij}^I - d_j^I \lambda^I_i \geq 0$$

The dual descent procedure for variables $\lambda^I_i$ is based on this observation, and follows the work of Saldanha da Gama (2002).

Dual Descent Procedure for Variables $\lambda^I_i$

1. $t \leftarrow 1$;
2. $i \leftarrow 1$;
3. if $Q_i - \sum_{j \in J} d_j^I < 0$, then go to 6.
4. $\delta \leftarrow \min \left\{ \lambda^I_i, \min_{j \in J} \left\{ \frac{v_j^I - c_{ij}^I - d_j^I \lambda^I_i}{d_j^I} \right\} \right\}$; If $\delta = 0$, then go to 6.
5. $SA_{i\tau}^\xi \leftarrow SA_{i\tau}^\xi + \delta \left( Q_i - \sum_{j \in J} d_j^I \right)$, $SR_{i\tau}^\xi \leftarrow SR_{i\tau}^\xi + \delta \left( Q_i - \sum_{j \in J} d_j^I \right)$, for $\tau, \xi$ such that $\tau \leq t \leq \xi$, $\lambda^I_i \leftarrow \lambda^I_i - \delta$;
6. $i \leftarrow i + 1$;
7. If $i > M$, then $t \leftarrow t + 1$. Else go to 3.
8. If \( t > T \), then stop. Else go to 2.

---

### 5.1.1

**Primal Procedure**

The primal procedure is very similar to the one described for DLPOCR. As a matter of fact, a solution is first built using the same procedure defined in Dias et al., 2002. After the execution of that procedure, the solution’s admissibility is tested (it may be not admissible due to the capacity restrictions). If the open facilities at any time period are not sufficient to serve the clients’ total demand, more services need to be open.

Consider the following definitions:

\[
I^* = \{ (i, \tau, \xi); S_{i\tau}^\xi = 0 \}
\]

\[
I_i^t = \{ i : (i, \tau, \xi) \in I^* \text{ and } \tau \leq t \leq \xi \}
\]

\[
I_i^+ = \{ i : \text{facility } i \text{ is open during period } t \}
\]

\[
I_{a_i}^+ = \{ (i, \tau, \xi): a_{i\tau}^\xi = 1 \}
\]

\[
I_{r_i}^+ = \{ (i, \tau, \xi): r_{i\tau}^\xi = 1 \}
\]

\[
F_i^t = \text{smallest cost incurred by having } i \notin I_i^+ \text{ operating during period } t.
\]

The primal procedure functioning scheme is as follows:

**C1-DLPOCR Primal Procedure**

1. Execute the DLPOCR primal procedure, building sets \( I_i^+, I_{a_i}^+, I_{r_i}^+ \).
2. \( t \leftarrow 1; \)
3. \( D \leftarrow \sum_{j} d_{ij}; C \leftarrow \sum_{i \in I_i^+} Q_i. \text{ If } D \leq C \text{ then go to 8.} \)
4. Calculate \( F_i^t \forall i \notin I_i^+ \).
5. Calculate \( F_i^t = \frac{F_i^t}{Q_i} \left[ \frac{Q_i}{\phi_i} \right], \forall i \notin I_i^+, \) where \( \phi_i = \begin{cases} D-C, & \text{if } C+Q_i < D \\ Q_i, & \text{otherwise} \end{cases} \).
6. Consider \( i' \) such that \( F_i^t = \min_{i \notin I_i^+} F_i^t \);
7. \( I_i^+ = I_i^+ \cup \{i'\}; \text{ Rebuild sets } I_{a_i}^+, I_{r_i}^+ \text{ and } I_i^+, \forall t; C \leftarrow C + Q_i; \text{ If } D \leq C \text{ then go to 8. Else go to 5.} \)
8. \( t \leftarrow t + 1; \) If \( t \leq T \) go to 3. Else go to 9.

9. \( t \leftarrow 1; \)

10. Solve one transportation problem considering as sources the set \( J \) of clients (with supplies \( d_{jt} \)), as destinations the set \( I_{it}^+ \) (with demands \( Q_i \)), and transportation costs (per unit) given by \( c_{ij}^t / d_{jt}^t \).

11. \( t \leftarrow t + 1; \) If \( t \leq T \) go to 10. Else stop.

Step 5 will penalize all services whose capacity is not sufficient to cover the difference between the clients’ total demand and the total capacity of open services (in this case it will be necessary to open at least one more facility).

The calculation of \( F_i^t \) for all \( i \notin I_{it}^+ \) is not trivial, especially if facility \( i \) belongs to one or more sets \( I_{it}^+ \), \( i \neq t \). All hypotheses need to be tested (considering the insertion of a new variable in sets \( I_A^+ \) or \( I_R^+ \), and the possible deletion of one or more variables from these sets). Consider that facility \( i \) is not open during period \( t \) but is open in time periods before and after \( t \), as depicted in figure 1.

\[
\begin{align*}
\text{Time periods } a, b, c, d \text{ can be defined formally as:} \\
b & = \max \left\{ 0, \max_{t' < t} \left\{ t' : i \in I_{it}^+ \right\} \right\}; \quad a = \left\{ t' : (i, t', b) \in I_A^+ \cup I_R^+ \right\}; \\
c & = \min \left\{ T + 1, \min_{t' > t} \left\{ t' : i \in I_{it}^+ \right\} \right\}; \quad d = \left\{ t' : (i, c, t') \in I_A^+ \cup I_R^+ \right\};
\end{align*}
\]

Time period \( b \) represents the time period less than and nearest to \( t \) such that facility \( i \) is operating. Time period \( c \) represents the time period greater than and nearest to \( t \) such that facility \( i \) is operating.
Calculation of $F_i^t$ for $i \notin I_i^+$

1. If $b = 0$ and $c = T + 1$ then $F_i^t \leftarrow \min \{FA_{it}^b : 1 \leq t \leq \xi \leq T \}$; Stop.

2. If $b = 0$ and $c \leq T$ then
   \[ F_i^t \leftarrow \min \{\min \{FA_{ia}^T - FA_{id}^T : t \leq \xi \leq T\} : 1 \leq t \leq \xi \leq c \} \min \{FA_{ia}^T - FA_{ic}^T : 1 \leq t \leq \xi \}, \]
   Stop.

3. If $b > 0$ and $c = T + 1$ then go to 4. Else go to 7.

4. If $(i, a, b) \in I_A^+$ then go to 5. Else go to 6.

5. $F_i^t \leftarrow \min \{\min \{FA_{ia}^T - FA_{ia}^b : t \leq \xi \leq T\} : 1 \leq t \leq \xi \leq c \}$.
   Stop.

6. $F_i^t \leftarrow \min \{\min \{FR_{ia}^T - FR_{ia}^b : t \leq \xi \leq T\} : 1 \leq t \leq \xi \leq c \}$.
   Stop.

7. If $(i, a, b) \in I_A^+$ then go to 8. Else go to 9.

8. $F_i^t \leftarrow \min \{\min \{FA_{ia}^T - FA_{ia}^d - FR_{ic}^d, \min \{FA_{ia}^T - FA_{ia}^b, t \leq \xi \leq c \} \}, \min \{FA_{ia}^T - FA_{ia}^b, t \leq \xi \leq c \} \min \{FR_{ia}^T - FR_{ic}^d, b < t \leq \xi \}, \)
   \min \{FR_{ia}^T - FR_{ia}^b, b < t \leq \xi \}, \min \{FR_{ia}^T - FR_{ic}^d, b < t \leq \xi \}, \)
   Stop.

9. $F_i^t \leftarrow \min \{\min \{FR_{ia}^T - FR_{ia}^b - FR_{ic}^d, \min \{FR_{ia}^T - FR_{ia}^b, t \leq \xi \leq c \} \}, \min \{FR_{ia}^T - FR_{ia}^b, t \leq \xi \leq c \} \min \{FR_{ia}^T - FR_{ic}^d, b < t \leq \xi \}, \)
   Stop.

Step 1 considers the situation where there is no $(i, \tau, \xi) \in I_A^+ \cup I_R^+$. In this case the procedure looks for variable $a_{it}^b$, $\tau \leq t \leq \xi$, with the smallest fixed cost. Step 2 considers the situation where there is no $(i, \tau, \xi) \in I_A^+ \cup I_R^+$, with $\xi < t$, but there is $(i, c, d) \in I_A^+$. In this case, the smallest cost of including facility $i$ in set $I_i^+$ will be given by the smallest cost between:

1. considering the smallest fixed cost of variables $a_{it}^b$, $\tau \leq t \leq \xi < c$, plus the cost of changing variable $a_{ic}^d$ from one to zero and variable $r_{ic}^d$ from zero to one;
2. considering the smallest fixed cost of variables $a_{it}^d$, $\tau \leq t$, minus the fixed cost of variable $a_{ic}^d$ that will be changed from one to zero.

Steps 4-6 consider the situation in which $(i, a, b) \in I_A^+ \cup I_R^+$, but there is no $(i, \tau, \xi)$, $t < \tau \leq \xi$. If $(i, a, b) \in I_A^+$, then $F_i^t$ will be calculated as the smallest cost between:

1. the smallest fixed cost of variables $a_{ia}^T$, $\xi \geq t$, minus the fixed cost of variable $a_{ia}^b$ that is changed from one to zero;
2. the smallest fixed cost of variables \( r_{i \tau}^\xi \), \( b < \tau < t \leq \xi \).

If \((i, a, b) \in I_R^+\), then \( F_i^j \) will be calculated as the smallest cost between:

1. the smallest fixed cost of variables \( r_{ia}^\xi \), \( \xi \geq t \), minus the fixed cost of variable \( r_{ia}^b \), that is changed from one to zero;
2. the smallest fixed cost of variables \( r_{i \tau}^\xi \), \( b < \tau < t \leq \xi \).

Steps 7-9 consider the situation when there is \((i, a, b) \in I_A^+ \cup I_R^+\), and also \((i, c, d) \in I_A^+ \cup I_R^+\). If \((i, a, b) \in I_A^+ \) (\((i, a, b) \in I_R^+\)) then \( F_i^j \) will be calculated as being the smallest cost between:

1. the fixed cost of variable \( a_{ia}^d \left( r_{ia}^d \right) \), minus the fixed costs of variables \( a_{ia}^b \left( r_{ia}^b \right) \) and \( r_{ic}^d \) that will be changed from one to zero;
2. the smallest fixed cost of variables \( r_{i \tau}^\xi \), \( b < \tau \leq t \leq \xi < c \).
3. the smallest fixed cost of variables \( a_{ia}^d \left( r_{ia}^d \right) \), \( t \leq \xi < c \), minus the fixed cost of variable \( a_{ia}^b \left( r_{ia}^b \right) \) that is changed from one to zero.
4. the smallest fixed cost of variables \( a_{i \tau}^d \left( r_{i \tau}^d \right) \), \( b < \tau \leq t \), minus the fixed cost of variable \( r_{ic}^d \) that will be changed from one to zero.

In step 7 of the C1-DLPOCR primal procedure, after choosing the facility \( i \) that will be open during period \( t \), the sets \( I_A^+, I_R^+ \) and \( I_t^+, \forall t \), are changed according to the calculation of the corresponding \( F_i^j \).

In Steps 9-11 of the C1-DLPOCR primal procedure, \( T \) transportation problems are solved that calculate the optimal value of the assignment variables \( x_{ij}^t \), given the fixed set \( I_t^+ \) of open facilities in each time period.

Example

Consider the following example, with three time periods, three facilities and five clients. The clients’ demands are shown in table 1. The fixed costs and maximum capacities are shown in table 2, and the total transportation costs are shown in table 3.
Table 1 – Clients’ demands $d^t_j$

<table>
<thead>
<tr>
<th></th>
<th>$FA^1_i$</th>
<th>$FA^2_i$</th>
<th>$FA^3_i$</th>
<th>$FA^1_j$</th>
<th>$FA^2_j$</th>
<th>$FA^3_j$</th>
<th>$FR^1_j$</th>
<th>$FR^2_j$</th>
<th>$FR^3_j$</th>
<th>capacity</th>
</tr>
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<td>154</td>
<td>157</td>
<td>130</td>
<td>133</td>
<td>157</td>
<td>113</td>
<td>114</td>
<td>145</td>
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</tr>
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<td>40</td>
<td>45</td>
<td>60</td>
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<td>52</td>
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<td>15</td>
</tr>
<tr>
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<td>29</td>
<td>50</td>
<td>47</td>
<td>50</td>
<td>12</td>
<td>14</td>
<td>14</td>
<td>58</td>
<td>19</td>
</tr>
</tbody>
</table>

Table 2 – Fixed Costs and Capacities

Period $t = 1$

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tbody>
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<td>47</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>25</td>
<td>19</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>52</td>
<td>28</td>
<td>18</td>
<td>25</td>
</tr>
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</table>

Period $t = 2$

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<th>5</th>
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<tbody>
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<td>7</td>
<td>50</td>
<td>14</td>
<td>14</td>
</tr>
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<td>15</td>
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<td>14</td>
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<td>5</td>
<td>29</td>
<td>17</td>
<td>21</td>
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</table>

Period $t = 3$

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<th>5</th>
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<td>45</td>
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<td>15</td>
<td>1</td>
<td>19</td>
<td>12</td>
<td>16</td>
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<td>30</td>
<td>5</td>
<td>27</td>
<td>18</td>
<td>19</td>
</tr>
</tbody>
</table>

Table 3 – Total transportation costs $c^t_{ij}$

Dual variables $v^t_j$ (as described in Dias et al, 2002) are initialized as follows:
After the execution of the dual ascent procedure for variables $v_j^t$, they became:

\[
\begin{array}{cccccc}
 t & j & 1 & 2 & 3 & 4 & 5 \\
 1 & 13 & 25 & 19 & 12 & 16 \\
 2 & 15 & 2 & 20 & 12 & 14 \\
 3 & 15 & 1 & 19 & 12 & 14 \\
\end{array}
\]

And the slacks:

\[
\begin{array}{cccccccc}
 i & SA_{i1} & SA_{i2} & SA_{i3} & SA_{i1}^2 & SA_{i2}^2 & SA_{i3}^2 & SR_{i1}^2 & SR_{i2}^3 \\
 1 & 100 & 154 & 155 & 130 & 131 & 155 & 113 & 112 & 143 \\
 2 & 0 & 0 & 0 & 43 & 50 & 40 & 46 & 42 & 20 \\
 3 & 20 & 29 & 50 & 47 & 50 & 12 & 14 & 14 & 58 \\
\end{array}
\]

This dual solution has an objective function value equal to 251. The primal procedure considers variable $a_{21}^3$ equal to one. As the total clients’ demand is greater than the total available capacity in each time period, it is needed to open more services. The procedure chooses to open services 1 and 3 from period 1 to 3 ($a_{11}^3 = 1; a_{31}^3 = 1$). This solution has an objective function value equal to 474.18.

The procedure that first changes the dual solution is the dual ascent procedure for variables $\lambda_j^t$. Consider variable $\lambda_2^3$. The procedure begins by calculating the $\delta$ value as:

\[
\delta \leftarrow \max \left\{ \frac{1}{5} \max \{0,17 - 15\}, \frac{1}{2} \max \{0,5 - 1\}, \frac{1}{12} \max \{0,21 - 19\}, \frac{1}{5} \max \{0,12 - 12\}, \frac{1}{10} \max \{0,16 - 16\} \right\} = 2
\]

The set $J(2)$ is equal to set $J$. $\Omega(2) = 2 + 4 + 2 - 2 \times 15 = -22$. This value is less than zero, so $\delta'$ is changed to 2 and $\delta$ will now be equal to 0.4. The set $J(0.4)$ is equal to $\{1, 3, 4, 5\}$.

---

4 The procedure begins by opening service 3 and 1 (in this order) only in period one, then from period one to two and finally opens service 3 from period one to the end of the planning horizon.
and \( \Omega(0.4) = 2 + 2 + 0.4 \times 2 - 0.4 \times 15 = -1.2 \), that is still less than zero. The procedure changes \( \delta' \) to 0.4, and \( \delta \) will now be equal to \( \frac{1}{6} \), with the corresponding set \( J\left(\frac{1}{6}\right) \) equal to \( \{3,4,5\} \), with \( \Omega\left(\frac{1}{6}\right) = 2 + \frac{1}{6} \times 5 + \frac{1}{6} \times 2 - \frac{1}{6} \times 15 = \frac{2}{3} = 0.6(6) \). This value is greater than zero, which means that variable \( \lambda^3_2 \) will be increased by \( \frac{1}{6} \) and slacks are changed:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( SA^1_1 )</th>
<th>( SA^2_1 )</th>
<th>( SA^3_1 )</th>
<th>( SA^2_2 )</th>
<th>( SA^3_2 )</th>
<th>( SA^3_3 )</th>
<th>( SR^2_2 )</th>
<th>( SR^3_2 )</th>
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</tr>
</thead>
<tbody>
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<td>0</td>
<td>0.67</td>
<td>43</td>
<td>50.67</td>
<td>40.67</td>
<td>46</td>
<td>42.67</td>
<td>20.67</td>
</tr>
</tbody>
</table>

This change allows the dual ascent procedure to increase the values of dual variables \( v^3_1 \) and \( v^3_5 \) to 17.67 and also \( v^3_4 \) to 12.83, increasing the dual objective function value. After the execution of the dual ascent procedure for variables \( v'_{j} \), slacks become:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( SA^1_1 )</th>
<th>( SA^2_1 )</th>
<th>( SA^3_1 )</th>
<th>( SA^2_2 )</th>
<th>( SA^3_2 )</th>
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<td>1</td>
<td>100</td>
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<td>152.67</td>
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<td>46</td>
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<tr>
<td>3</td>
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<td>47</td>
<td>50</td>
<td>12</td>
<td>14</td>
<td>14</td>
<td>58</td>
</tr>
</tbody>
</table>

The dual ascent procedure for variables \( \lambda^3_i \) changes the dual solution again in the next iteration of the heuristic, changing \( \lambda^3_1 \) to 0.13. Slacks are changed but variables \( v'_{j} \) cannot be increased:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( SA^1_1 )</th>
<th>( SA^2_1 )</th>
<th>( SA^3_1 )</th>
<th>( SA^2_2 )</th>
<th>( SA^3_2 )</th>
<th>( SA^3_3 )</th>
<th>( SR^2_2 )</th>
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<tr>
<td>1</td>
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<td>12</td>
<td>14</td>
<td>14</td>
<td>58</td>
</tr>
</tbody>
</table>
The dual descent procedure for variables $\lambda^t_i$ doesn’t change the dual solution. Consider, for instance, variable $\lambda^3_1$. The procedure calculates $\sum_{j \in J} d^3_j = d^3_1 + d^3_3 = 15$. As $Q_1$ is equal to 13, then $13 - 15$ is less than zero, so the dual variable is not changed.

Imagine, however, that $Q_1$ is equal to 20. In this situation, the procedure would calculate the $\delta$ value as:

$$\delta \leftarrow \min \left\{ \frac{5 - 8.3}{2}, \frac{21 - 46.7}{12}, \frac{-12 - 14.7}{5} \right\} = 0.13$$

The dual variable would be decreased to zero, and the slacks would take the values they had before the dual ascent procedure.

**Maximum and Minimum Capacities**

As is easily seen by expressions (28) and (29), dual variables $\beta^t_i$ behave in a symmetric way when compared to the dual variables $\lambda^t_i$. Therefore, the dual ascent procedure for one variable becomes the dual descent procedure for the other, and the other way around.

The only difference between the primal-dual heuristic described for C1-DLPOCR and the primal-dual heuristic build for C2-DLPOCR has to do with the construction of admissible primal solutions. Given sets $I^+_t$, a solution is admissible if:

$$\sum_{i \in I^+_t} Q'_i \leq \sum_{j} d^t_j, \quad \forall t \tag{41}$$

$$\sum_{j} d^t_j \leq \sum_{i \in I^+_t} Q_i, \quad \forall t \tag{42}$$

For a given period $t$ only one of these conditions can be violated.

**Primal Procedure**

To build an admissible solution to C2-DLPOCR, it is first guaranteed the satisfaction of (42). If conditions (41) are violated, then the solution is changed by closing some open facilities or exchanging open facilities with closed facilities with smaller minimum capacities.

Consider:

$f^t_i =$ smallest cost incurred by closing $i \in I^+_t$ during period $t$. 

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$g_i^t = \text{smallest cost incurred by closing } i \in I_t^+ \text{ during period } t \text{ and opening one or more services } i' \not\in I_t^+ \text{ such that the total maximum capacity of services } i' \text{ is sufficient to satisfy condition (42) and the total minimum capacity of services } i' \text{ is less than } Q'_i$. The set of services $i' \not\in I_t^+$ that are open is designated by $R(i,t)$.

**C2-DLPOCR Primal Procedure**

1. Execute the DLPOCR primal procedure, building sets $I_t^+$, $I_A^+$, $I_R^+$.
2. $t \leftarrow 1$;
3. $D \leftarrow \sum_j q_j^t$; $C_{\text{max}} \leftarrow \sum_i q_i$; $C_{\text{min}} \leftarrow \sum_i q'_i$. If $C_{\text{min}} \leq D \leq C_{\text{max}}$ go to 16. Else go to 4.
4. Calculate $F_i^t, \forall i \not\in I_t^+$. $F_i^t \leftarrow F_i^t \cdot (1 + Q_i'/Q_i)$, $\forall i \not\in I_t^+$. If $D \leq C_{\text{max}}$ then go to 8. Else go to 5.
5. Calculate $F_i^t = \frac{F_i^t}{Q_i} \cdot \phi_i, \forall i \not\in I_t^+$, where $\phi_i = \begin{cases} D - C_{\text{max}}, & \text{if } C_{\text{max}} + Q_i < D \\ Q_i, & \text{otherwise} \end{cases}$.
6. Consider $i'$ such that $F_{i'}^t = \min_{i \not\in I_t^+} \{F_i^t\}$;
7. $I_t^+ = I_t^+ \cup \{i'\}$; Rebuild sets $I_A^+$ and $I_R^+$, $\forall t$; $C_{\text{max}} \leftarrow C_{\text{max}} + Q_i'$; $C_{\text{min}} \leftarrow C_{\text{min}} + Q'_i$; If $D \leq C_{\text{max}}$ then go to 8. Else go to 5.
8. If $C_{\text{min}} \leq D$ then go to 16. Else go to 9.
9. Calculate $f_i^t, \forall i \in I_t^+$.
10. If $\{i \in I_t^+ : C_{\text{max}} - Q_i \geq D\} = \emptyset$ then go to 12. Else consider $i'$ such that $f_{i'}^t = \min_{i \in I_t^+} \{f_i^t \cdot \phi_i\}$, where $\phi_i = \begin{cases} 1 + \frac{C_{\text{min}} - D}{C_{\text{min}} - D}, & \text{if } f_i^t > 0 \\ 1, & \text{otherwise} \end{cases}$.
11. $I_t^+ = I_t^+ \setminus \{i'\}$; Rebuild sets $I_A^+$ and $I_R^+$; $C_{\text{min}} \leftarrow C_{\text{min}} - Q'_i$; $C_{\text{max}} \leftarrow C_{\text{max}} - Q_i$; If $C_{\text{min}} \leq D$ then go to 16. Else go to 12.
12. Calculate $g_i^t, \forall i \in I_t^+$. If $g_i^t = +\infty, \forall i \in I_t^+$, stop: it is not possible to find an admissible solution.
13. Consider $i'$ such that $g_{i'}^t = \min_{i \in I_t^+} \{g_i^t\}$. $I_t^+ = I_t^+ \setminus \{i'\}; I_t^+ = I_t^+ \cup I(i', t)$.
14. Rebuild sets \( I^+_A, I^+_R, I^+_{i}, \forall i; C_{\text{min}} \leftarrow C_{\text{min}} - Q'_t + \sum_{i \in I(t', t)} Q'_i; C_{\text{max}} \leftarrow C_{\text{max}} - Q'_t + \sum_{i \in I(t', t)} Q_i \).

15. If \( C_{\text{min}} \leq D \) then go to 16. Else go to 12.

16. \( t \leftarrow t + 1; \) If \( t \leq T \) go to 3. Else go to 17.

17. \( t \leftarrow 1; \)

18. Solve one transportation problem considering as sources the set \( J \) of clients (with supplies \( d_j \)), two destinations \((i_1, i_2)\) for each \( i \in I^+_{i} \) (one with demand \( Q'_i \) and the other with demand \( Q_i - Q'_i \)), and transportation costs (per unit) given by \( c_{ij}' / d_j' \). If there is the need to consider a fictitious source, the transportation costs from this source to the destinations \( i \) should be considered +\( \infty \) (in order to guarantee that constraints (25) are satisfied).

19. \( t \leftarrow t + 1; \) If \( t \leq T \) go to 19. Else stop.

Some of the steps of this procedure require further explanations. In step 4, \( F^t_i \) is calculated as indicated in C1-DLPOCR primal procedure, but guaranteeing that constraints (41) remain admissible for all periods \( t' < t \). It is then changed, so that the relation between the maximum and minimum capacity of each facility is taken into account. A facility that has a minimum capacity near its maximum capacity will, more likely, cause a violation of (41).

In step 9, the calculation of \( f^t_i \) is done in the following way: if \( i \in I^+_{i} \), then there exists \((i, a, b) \in I^+_A \cup I^+_R\) such that \( a \leq t \leq b \).

If \((i, a, b) \in I^+_A\) then: 
\[
 f^t_i = \begin{cases} 
 -F^b_A, & \text{if } a = b \\
 F^b_A - F^b_{t+1} + F^b_A, & \text{otherwise}
\end{cases}
\]

If \((i, a, b) \in I^+_R \) then: 
\[
 f^t_i = \begin{cases} 
 -F^b_R, & \text{if } a = b \\
 F^b_R - F^b_{t+1} + F^b_R, & \text{otherwise}
\end{cases}
\]

In step 10, the procedure tries to close facilities whose maximum capacity is not necessary to guarantee the satisfaction of conditions (42). All facilities such that its closure is not sufficient to satisfy conditions (41) are penalized. The procedure will reach step 12 whenever it is not possible to satisfy simultaneously (41) and (42) by insertion or deletion of facilities from set \( I^+_{i} \). If this happens, it will be necessary to change open facilities by closed facilities. When closing a facility, the total maximum capacity available at time period \( t \) will diminish. This makes it necessary to open one or more facilities (otherwise conditions (42) would be violated).
In the calculation of $g_i^t$, it is necessary to take into account the fixed cost of the service closed but also the fixed costs of services that will be open.

**Calculation of $g_i^t$, $i \in I_t^+$**

1. $Dif' \leftarrow D - (C_{max} - Q_i), g_i^t \leftarrow f_i^t; I(i, t) = \emptyset; Cap \leftarrow 0;$

2. Calculate $F_i^{t'} = \frac{F_i^t}{Q_i} \cdot \phi_i, \forall i \in I_t^+$, where $\phi_i = \begin{cases} Dif', & \text{if } Q_i < Dif' \\ Q_i, & \text{otherwise} \end{cases}$.

3. If $\exists i' \not\in I_t^+: Cap + Q_i' < Q_i'$, then $g_i^t \leftarrow +\infty$, stop. Else go to 4.

4. Consider $i' \not\in I_t^+$ such that $F_i^{t'} = \min_{l \in I_t^+ \cup I(i, t), Cap + Q_i' < Q_i'} \left\{ F_i^{t'} \right\}$.

5. $I(i, t) = I(i, t) \cup \{i'\}; g_i^t \leftarrow g_i^t + F_i^{t'}; Dif \leftarrow Dif - Q_i; Cap \leftarrow Cap + Q_i'$.

6. If $Dif' \leq 0$ stop; else go to 2.

The C2-DLPOCR primal procedure doesn’t guarantee the calculation of an admissible solution. If, in step 12, $g_i^t = +\infty, \forall i \in I_t^+$, it will not be possible to build an admissible solution ($g_i^t = +\infty$ if the procedure cannot build a set $I(i, t)$ such that the total minimum capacity of facilities belonging to $I(i, t)$ is less than $Q_i'$). Although this situation can occur, it has not been a problem in the computational tests performed.

**Example**

Consider a problem with three services with maximum capacities equal to 9, 20, 5 and minimum capacities equal to 8, 5 and 1, respectively. The total clients’ demand is equal to 15. Suppose that at step 3 of the primal procedure for some period $t$ only service 3 is operating. This means that $C_{max}$ is equal to 5, $C_{min}$ is equal to 1 and $D$ is equal to 15. Consider that $F_{1}^{t} = 5$ and $F_{2}^{t} = 7$. Then, at step 4, these values are transformed: $F_{1}^{t} = 5 \times \left(1 + \frac{8}{9}\right) = 8.44$; $F_{2}^{t} = 7 \times \left(1 + \frac{5}{20}\right) = 8.75$. Step 5 calculates $F_{1}^{t'} = \frac{8.44}{9} \times \left[\frac{15 - 5}{9}\right] = 1.875$ and $F_{2}^{t'} = \frac{8.75}{20} = 0.44$. The procedure chooses service 2. As $C_{min} \leq D \leq C_{max}$, this solution is admissible for period $t$.  

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Let us consider the same example, changing only the capacities of service 3 (minimum capacity equal to 20 and maximum capacity equal to 30) and also the maximum capacity of service 2 (equal to 10). In this case the solution is not admissible because $C_{\text{min}}$ is greater than $D$. It is not possible to close any open service during period $t$ (because there is no open service $i$ such that $C_{\text{max}} - Q_i \geq D$). The procedure will try to exchange the open service by one or more closed services, guaranteeing the satisfaction of the maximum capacity restrictions and decreasing the total minimum capacities of open services during period $t$. Suppose that $f^{t}_{3}$ is equal to $-10$:

$$D_{\text{if}} = 15 - (30 - 30) = 15; \quad F^{t}_{1} = \frac{5}{9} \times \left\lceil \frac{15}{9} \right\rceil = 0.93; \quad F^{t}_{2} = \frac{7}{10} = 0.7.$$  

For both services 1 and 3, $Q^{t}_{i} < Q^{t}_{3}$, so the procedure chooses service 2. $I(3,t) = \{2\}; \quad D_{\text{if}} = 5$ and $Cap = 5$. The procedure goes back to step 2, and calculates $F^{t}_{1} = \frac{5}{9} = 0.56$. As $5 + 8 = 13$ is less than 20, then $I(3,t) = \{2,1\}$ and $D_{\text{if}}$ is equal to $-4$. As $D_{\text{if}}$ is less than zero, the procedure terminates. The cost of exchanging service 3 with services 1 and 2 is given by $-10 + 0.93 + 0.56$. As service 3 is the only open service, it will be closed during period $t$, and services 1 and 2 are going to be open. This solution is already admissible for period $t$.

**Maximum Decreasing Capacities**

This problem is the hardest one to solve. The behavior of the dual variables $\lambda^{t}_{i}$ is different from what has been described for the previous problems and is more difficult to find an admissible primal solution. In the previous problems, after deciding which facilities are open at each time period, the optimal value of the assignment variables could be calculated through the resolution of $T$ transportation problems. In the present problem, the resolution of $T$ transportation problems doesn’t guarantee the calculation of the optimal assignments of clients to facilities. The resolution of the following linear programming problem does.

**PL1**

$$\text{Min} \sum_{t} \sum_{i} \sum_{j} c^{t}_{ij} x^{t}_{ij} \quad (43)$$

subject to:
\[
\sum_{i \in I_t^+} \chi_{ij}^t = 1, \quad \forall j, t \tag{44}
\]

\[
Q_t \sum_{\tau = 1}^T \left( a_{i\tau}^e + r_{i\tau}^e \right) - \sum_{\tau = 1}^T \sum_j d_j^s x_{ij}^t \geq 0, \quad \forall t, i \in I_t^+ \tag{45}
\]

\[
\chi_{ij}^t \geq 0, \forall j, t, i \in I_t^+.
\]

**Dual Ascent Procedure for variables \( \lambda_t^i \)**

Variable \( \lambda_t^i \) influences the value of all slacks \( SA_{i\tau}^r \) and \( SR_{i\tau}^r, \tau \leq t \). Consider the following definitions:

\[
\Delta = \max_{j \in J, \tau \leq t} \left\{ \frac{1}{d_j^s} \max_{\xi = \tau} \left\{ 0, v_j^s - c_{ij}^s - \sum_{\xi = \tau}^T a_{ij}^e \right\} \right\};
\]

\[
J(t) = \left\{ j \in J : v_j^t - c_{ij}^t - \sum_{\xi = t}^T d_j^s \lambda_t^i \leq \delta \cdot d_j^s \right\}. \tag{46}
\]

**Proposition 3:** If variable \( \lambda_t^i \) is increased by \( \delta \in ]0, \Delta] \), then slacks \( SA_{i\tau}^r \) and \( SR_{i\tau}^r, \tau \leq t \), will be changed by:

\[
\Phi(\delta, \tau, \xi) = \min_{\xi, t} \left( \sum_{s = t} J(t) \left\{ \max_{j \in J(s)} \left\{ 0, v_j^s - c_{ij}^s - \sum_{\psi = s}^T a_{ij}^e \right\} + \sum_{j \notin J(s)} \delta \cdot d_j^s \right\} - \delta \cdot Q_t \right) \tag{47}
\]

**Proof:**

If \( \lambda_t^i \) is increased by \( \delta \in ]0, \Delta] \), all sums \( \sum_{j = \tau}^T d_j^s \lambda_t^i \), with \( s \leq t \), will be increased by \( d_j^s \delta \). These sums influence the values of all slacks \( SA_{i\tau}^r \) and \( SR_{i\tau}^r \), with \( \tau \leq t \). If \( t > \xi \) then all sums with \( \tau \leq s \leq \xi \) have to be taken into account. If \( t < \xi \) then only sums with \( \tau \leq s \leq t \) will change (sums with \( s > t \) will not be altered).

---

\(^5\) Variables \( a_{i\tau}^e \) and \( r_{i\tau}^e \) are fixed to one or zero.
For each $s$, $\tau \leq s \leq \min\{\xi, t\}$, $v^s_j - c^s_{ij} - \sum_{\psi=s}^{T} d^s_j \lambda^\psi_i$ with $j \in J_1(s)$ will become less than or equal to zero (and the corresponding $w^s_{ij}$ variable will be equal to zero). For all $j \not\in J_1(s)$, variables $w^s_{ij}$ will be decreased by $d^s_j \delta$.

Dual variable $\lambda^l_i$ influences all slacks $SA^\xi_{i\tau}$ and $SR^\xi_{i\tau}$, $\tau \leq t$, not only due to sums $\sum_{\psi=s}^{T} d^s_j \lambda^\psi_i$, with $\tau \leq s \leq t$, but also due to the sum $Q_i \sum_{\psi=\tau}^{T} \lambda^\psi_i$. This sum will be increased by $\delta Q_i$.

Therefore, it can be concluded that the total change in slacks $SA^\xi_{i\tau}$ and $SR^\xi_{i\tau}$, $\tau \leq t$, due to a change $\delta$ in dual variable $\lambda^l_i$ is given by $\Phi(\delta, \tau, \xi)$.

As can be seen by expression (47), slacks influenced by the increase in the dual variable will have different behaviors: some can be increased while others can be decreased.

**Proposition 4:** Consider $\lambda^l_i$ is increased by $\delta^*$, with $\delta^* > \Delta$, being the resulting slacks $SA^\xi_{i\tau}$ and $SR^\xi_{i\tau}$, with $\tau \leq t$. It is possible to find $\delta \in [0, \Delta]$ such that if $\lambda^l_i$ is increased by $\delta$ instead of $\delta^*$, the resulting values of all slacks $SA^\xi_{i\tau}$ and $SR^\xi_{i\tau}$, with $\tau \leq t$, will be greater than or equal to $SA^\xi_{i\tau}$ and $SR^\xi_{i\tau}$, with $\tau \leq t$.

**Proof:** Follows directly from proposition 3 and definition of $\Delta$.

Proposition 3 motivates the following dual ascent procedure for variables $\lambda^l_i$.

**Dual Ascent Procedures for Variables $\lambda^l_i$**

1. $t \leftarrow 1$;
2. $i \leftarrow 1$;
3. $\delta^* \leftarrow +\infty$;
4. $\delta \leftarrow \max_{j \in J} \left\{ \frac{1}{d_j} \max_{\tau \leq t} \left\{ 0, v_j^\tau - c_{ij}^\tau - \sum_{\xi=\tau}^T d_j^\tau \lambda_i^\xi \right\} \right\}$; If $\delta = 0$, then go to $9$.

$$\frac{1}{d_j} \max_{\tau \leq t} \left\{ 0, v_j^\tau - c_{ij}^\tau - \sum_{\xi=\tau}^T d_j^\tau \lambda_i^\xi \right\} < \delta'$$

9.

5. Compute $J_1(t)$ as in (46). If $\exists$ $SA_{i\tau}^\xi$ or $SR_{i\tau}^\xi$, $\tau \leq t$, such that $SA_{i\tau}^\xi + \Phi(\delta, \tau, \xi) < 0$ or $SR_{i\tau}^\xi + \Phi(\delta, \tau, \xi) < 0$ go to 6. Else go to 7.

6. If $\delta' = 0$, then go to 9. Else $\delta^* \leftarrow \delta$. Go to 4.

7. $\lambda_i^\tau \leftarrow \lambda_i^\tau + \delta$; $SA_{i\tau}^\xi \leftarrow SA_{i\tau}^\xi + \Phi(\delta, \tau, \xi)$ and $SR_{i\tau}^\xi \leftarrow SR_{i\tau}^\xi + \Phi(\delta, \tau, \xi)$, $\forall \tau \leq t$.

8. Execute dual ascent procedure for variables $v_j^\tau$.

9. $i \leftarrow i + 1$; if $i > M$ then go to 10. Else go to 3.

10. $t \leftarrow t + 1$; if $t > T$ then stop. Else go to 2.

5.1.2

Dual Descent Procedure for variables $\lambda_i^\tau$

A decrease in the dual variable $\lambda_i^\tau$ will increase all values $v_j^\tau - c_{ij}^\tau - \sum_{\xi=\tau}^T d_j^\tau \lambda_i^\xi$, $\forall \tau \leq t$.

Proposition 5: If $\lambda_i^\tau$ is decreased by $\delta$, with:

$$0 < \delta \leq \min_{\tau \leq t} \left\{ \frac{v_j^\tau - c_{ij}^\tau - \sum_{\xi=\tau}^T d_j^\tau \lambda_i^\xi}{d_j^\tau} \right\},$$

then all slacks $SA_{i\tau}^\xi$ and $SR_{i\tau}^\xi$, with $\tau \leq t$, will be changed by:

$$\Omega(\delta, \tau, \xi) = \delta \left\{ Q_{i\tau} - \sum_{\psi=\tau} \sum_{j \in J} d_j^\psi \right\},$$

$$\sum_{j \in J} d_j^\psi \geq 0$$
Proof: If \( \lambda_i \) is decreased by \( \delta \), all values \( v_j^s - c_{ij}^s - \sum_{\psi = s}^{T} d_{ji}^s \lambda_i^\psi \), \( s \leq t \), will be increased by \( \delta d_j^s \). To guarantee that \( v_j^s - c_{ij}^s - \sum_{\psi = s}^{T} d_{ji}^s \lambda_i^\psi < 0 \) will remain less than or equal to zero:

\[
v_j^s - c_{ij}^s - \sum_{\psi = s}^{T} d_{ji}^s \lambda_i^\psi + \delta d_j^s \leq 0 \iff \delta \leq -\frac{v_j^s - c_{ij}^s - \sum_{\psi = s}^{T} d_{ji}^s \lambda_i^\psi}{d_j^s}.
\]

Therefore, the upper limit defined by (48) guarantees that for all \( j \) and \( s \) such that \( v_j^s - c_{ij}^s - \sum_{\psi = s}^{T} d_{ji}^s \lambda_i^\psi < 0 \), this value will continue less than zero. For each slack \( SA_{i\tau}^\xi \) and \( SR_{i\tau}^\xi \), with \( \tau \leq t \), all values \( v_j^s - c_{ij}^s - \sum_{\psi = s}^{T} d_{ji}^s \lambda_i^\psi \geq 0 \), with \( s \leq \min\{\xi, t\} \), will be increased by \( d_j^s \delta \). Each of these slacks is also influenced by the decrease \( \delta Q_i \) in sum \( Q_i \sum_{\psi = \tau}^{T} \lambda_i^\psi \). Therefore \( \Omega(\delta, \tau, \xi) \) expresses the change occurred in slacks \( SA_{i\tau}^\xi \) and \( SR_{i\tau}^\xi \), with \( \tau \leq t \), due to a decrease \( \delta \) in dual variable \( \lambda_i^I \).

Proposition 5 motivates the following dual descent procedure for variables \( \lambda_i^I \).

**Dual Descent Procedure for Variables \( \lambda_i^I \)**

1. \( t \leftarrow 1; \)
2. \( i \leftarrow 1; \)
3. \( \delta \leftarrow \min \left\{ \lambda_i^I, \left\{ \begin{array}{l} v_j^s - c_{ij}^s - \sum_{\psi = s}^{T} d_{ji}^s \lambda_i^\psi \\ v_j^s - c_{ij}^s - \sum_{\psi = s}^{T} d_{ji}^s \lambda_i^\psi < 0 \end{array} \right\} \right\} \); If \( \delta = 0 \), then go to 7.
4. If \( SA_{i\tau}^\xi + \Omega(\delta, \tau, \xi) < 0 \) or \( SR_{i\tau}^\xi + \Omega(\delta, \tau, \xi) < 0 \), for some \( \tau \leq t \), then:
\[
\delta \leftarrow \min_{\tau \leq t, \Omega(\delta, \tau, \xi) < 0} \left\{ \frac{SA_{t \tau}^e}{\Omega(\delta, \tau, \xi)/\delta} - \frac{SR_{t \tau}^e}{\Omega(\delta, \tau, \xi)/\delta} \right\}.
\]

5. If \(\delta = 0\) go to 7. Else \(SA_{t \tau}^e \leftarrow SA_{t \tau}^e + \Omega(\delta, \tau, \xi)\) and \(SR_{t \tau}^e \leftarrow SR_{t \tau}^e + \Omega(\delta, \tau, \xi), \forall \tau \leq t\).

\[
\lambda_i^t \leftarrow \lambda_i^t + \delta.
\]

6. Execute dual ascent procedure for variables \(v_i^t\).

7. \(i \leftarrow i + 1; \) if \(i > M\) then go to 8. Else go to 3.

8. \(t \leftarrow t + 1; \) if \(t > T\) then stop. Else go to 2.

---

**Primal Procedure**

The primal procedure, here developed, guarantees the calculation of a primal admissible solution, if one exists for C3-PLDOCR. It begins by calculating a solution for PLDOCR. When it is necessary to open more services during a period \(t\), the procedure calculates the cost of opening facilities not in operation during that time period, but also the cost of opening or reopening facilities before period \(t\). Every time a facility is (re) open, its capacity is increased.

Consider the following notation:

- \(h_i^t = \) smallest cost incurred by opening a facility \(i \not\in I_i^+\) during period \(t\).
- \(p_i^t = \) smallest cost incurred by reopening a facility \(i \in I_i^+\) at the beginning of a period \(t' < t\).
- \(Cap_i^t = \) Maximum capacity of facility \(i\) at the beginning of time period \(t\).

**C3-DLPOCR Primal Procedure**

1. Execute the DLPOCR primal procedure, building sets \(I_i^+, I_{\alpha}^+, I_{\beta}^+\).

2. Solve problem PL1 optimally using a general solver. If PL1 has no admissible solutions, go to 3. Else stop.

3. \(t \leftarrow 1. \) \(Cap_i^1 \leftarrow Q_i, \forall i \in I_i^+\) and \(Cap_i^1 \leftarrow 0, \forall i \not\in I_i^+\).

4. \(D' \leftarrow \sum_{\tau \in I_i^+} d_{\tau}^j; C' \leftarrow \sum_{\tau \in I_i^+} \text{Cap}_i^1. \) If \(D' \leq C'\) then go to 9.

5. Calculate \(h_i^t, \forall i \not\in I_i^+\) and \(p_i^t, \forall i \in I_i^+. \) \(h_i^t \leftarrow +\infty, \forall i \in I_i^+\) and \(p_i^t \leftarrow +\infty, \forall i \not\in I_i^+. \)

6. Choose \(i'\) such that \(\min \{h_i^t, p_i^t\} = \min \{h_i^t, p_i^t\}. \)

7. Rebuild sets \(I_{\alpha}^+, I_{\beta}^+, I_i^+, \forall t\) and recalculate \(Cap_i^t, \forall i,\) and \(C\) according to the choice made in 6.
8. If \( D > C \) then go to 5. Else go to 9.

9. Solve one transportation problem considering as sources the set \( J \) of clients (with supplies \( d_j^t \)), as destinations the set \( I_i^+ \) (with demands \( Cap_i^t \)), and transportation costs (per unit) given by \( c_{ij}^t/d_j^t \). Consider the values of the transportation variables designated by \( x_{ij}^t \).

10. \( Cap_i^t \leftarrow Cap_i^t - \sum_{j \in J} x_{ij}^t \), \( \forall i \in I_i^+ \).

11. \( t \leftarrow t + 1; \) if \( t > T \), then go to 2. Else go 12.

12. If \( \exists (i, t, \xi) \in I_A^+ \cup I_B^+ \), then \( Cap_i^t \leftarrow Cap_i^{t-1} + Q_i \). Else \( Cap_i^t \leftarrow Cap_i^{t-1} \). Go to 4.

Calculation of \( h_i^t \) is made almost exactly as the calculation of \( F_i^t \), in the C1-DLPOCR primal procedure. The only modification refers to steps 5, 6, 8, and 9. The possibility of changing the value of variables \( a_{ia}^b \) or \( r_{ia}^b \) (with \( a \) and \( b \) defined as before in the C1-DLPOCR primal procedure) is considered only if facility \( i \) has remaining capacity greater than zero at the end of time period \( b \) (otherwise, even if the facility was operational during time period \( t \), it would not increase the total available capacity).

Calculation of \( h_i^t \) for \( i \notin I_i^+ \)

1. If \( b = 0 \) and \( c = T + 1 \) then \( h_i^t \leftarrow \min \{FA_{i \tau}^{\xi} : 1 \leq \tau \leq T, \xi \leq T \} \); go to 10.

2. If \( b = 0 \) and \( c \leq T \) then:
   \[
   h_i^t \leftarrow \min \{\min \{FA_{i \tau}^{\xi} - FA_{i \tau}^d + FR_{i \tau}^d : 1 \leq \tau \leq t, \xi \leq c\}, \min \{FA_{i \tau}^d - FA_{i \tau}^d : 1 \leq \tau \leq T\}\}. \quad \text{Go to 10.}
   \]

3. If \( b > 0 \) and \( c = T + 1 \) then go to 4. Else go to 7.

4. If \((i, a, b) \in I_A^+ \) then go to 5. Else go to 6.

5. If \( Cap_i^t > 0 \), then:
   \[
   h_i^t \leftarrow \min \{\min \{FA_{ia}^{\xi} - FA_{ia}^b : t \leq \xi \leq T\}, \min \{FR_{ia}^{\xi} : b < t \leq \xi \leq T\}\};
   \]
   Else \( h_i^t \leftarrow \min \{FR_{ia}^{\xi} : b < t \leq \xi \leq T\} \). Go to 10.

6. If \( Cap_i^t > 0 \) then:
   \[
   h_i^t \leftarrow \min \{\min \{FR_{ia}^{\xi} - FR_{ia}^b : t \leq \xi \leq T\}, \min \{FR_{ia}^{\xi} : b < t \leq \xi \leq T\}\};
   \]
   Else \( h_i^t \leftarrow \min \{FR_{ia}^{\xi} : b < t \leq \xi \leq T\} \). Go to 10.

7. If \((i, a, b) \in I_A^+ \) then go to 8. Else go to 9.
8. If $Cap_i > 0$ then 
$$h_i^t \leftarrow \min \left\{ FA_a^d - FA_a^b - FR_c^d, \min \left\{ FR^d_{\xi}, b \leq \tau \leq \xi \right\}, \min \left\{ FA_a^b - FA_a^d, \xi < c \leq t \right\} \right\}.$$ 
Else $h_i^t \leftarrow \min \left\{ \min \left\{ FR^d_{\xi}, b \leq \tau \leq \xi \right\}, \min \left\{ FR^d_{\xi}, b \leq \tau \leq \xi \right\} \right\}.$ Go to 10.

9. If $Cap_i > 0$ then 
$$h_i^t \leftarrow \min \left\{ FR^d_{a} - FR^b_{a} - FR^d_{c}, \min \left\{ FR^d_{\xi}, b \leq \tau \leq \xi \right\}, \min \left\{ FR^d_{\xi}, b \leq \tau \leq \xi \right\} \right\}.$$ 
Else $h_i^t \leftarrow \min \left\{ \min \left\{ FR^d_{\xi}, b \leq \tau \leq \xi \right\}, \min \left\{ FR^d_{\xi}, b \leq \tau \leq \xi \right\} \right\}.$ Go to 10.

10. If the value of $h_i^t$ was calculated considering the fixed cost of a variable $a_{i\tau}^\xi$ or $r_{i\tau}^\xi$ such that $b<\tau$, then $C_i^l \leftarrow Q_i + Cap_i$. Else $C_i^l \leftarrow Cap_i$.

11. $h_i^t \leftarrow \frac{\phi_i}{D - C}$, if $C + C_i^l < D$ 
$$\phi_i = \begin{cases} 0, & \text{otherwise} \end{cases}$$ 

If a service is already open during time period $t$, there is the possibility of increasing its available capacity by (re) opening the facility before time period $t$. This can be achieved either by splitting variables $a_{i\tau}^\xi$ or $r_{i\tau}^\xi$, with $\tau \leq t$, that are considered equal to one in the present primal solution or by considering new variables $a_{i\tau}^\xi$ or $r_{i\tau}^\xi$ such that $i \notin I_{i+}$, for $\tau \leq t'$, $\xi < t$. The calculation of $p_i^t$ takes all these possibilities into account.

**Calculation of $p_i^t$ for $i \in I_{i+}^+$**

1. $p_i^t \leftarrow \min \left\{ +\infty, \min_{(i,\xi)\in I_{i+}^+} \left\{ FA_i^+ + FR_i^\xi + FA_i^c \right\}, \min_{(i,\xi)\in I_{i+}^+} \left\{ FR_i^\xi + FR_i^c - FR_i^\xi \right\} \right\}.$

2. $t1 \leftarrow 1$;

3. If $i \in I_{i+}$ go to 6. Else go to 4.

4. $b = \max \left\{ 0, \max_{t' < 1} \left\{ t' \in I_{i+}^+ \right\} \right\}; \quad c = \min \left\{ t, \min_{t' > 1} \left\{ t' \in I_{i+}^+ \right\} \right\};$

5. If $b=0$, there exists $(i,a,d) \in I_{i+}^+$. Then $p_i^t \leftarrow \min \left\{ p_i^1, \min_{b<\tau\leq\xi\leq c} \left\{ FA_i^\xi \right\} + FR_i^d - FR_i^d \right\}.$ Else 
$$p_i^t \leftarrow \min \left\{ p_i^t, \min_{b<\tau\leq\xi\leq c} \left\{ FR_i^\xi \right\} \right\}.$$

35
6. \( t_1 \leftarrow t_1 + 1 \); If \( t_1 = t \) then go to 7. Else go to 3.

7. \( p^t_i \leftarrow \frac{p^t_i}{Q^t_i} \left[ \frac{\phi_i}{Q^t_i} \right] \), where \( \phi_i = \begin{cases} D - C, & \text{if } C + Q_i < D \\ Q_i, & \text{otherwise} \end{cases} \)

### 5.1.3 Example

Consider the example of section 5.1.4, with the following changes: \( FA^{3}_{21} = 140 \) and \( FR^{3}_{23} = 190 \). After the dual ascent procedure for variables \( v^j_i \), dual variables and slacks take the following values:

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
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<tr>
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<td>51</td>
<td>19</td>
<td>12</td>
<td>16</td>
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<td>2</td>
<td>17</td>
<td>2</td>
<td>20</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>8</td>
<td>36</td>
<td>18</td>
<td>19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( i )</th>
<th>( SA^{1}_{i1} )</th>
<th>( SA^{2}_{i1} )</th>
<th>( SA^{3}_{i1} )</th>
<th>( SA^{2}_{i2} )</th>
<th>( SA^{3}_{i2} )</th>
<th>( SA^{3}_{i3} )</th>
<th>( SR^{2}_{i2} )</th>
<th>( SR^{3}_{i2} )</th>
<th>( SR^{3}_{i3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>154</td>
<td>135</td>
<td>130</td>
<td>111</td>
<td>135</td>
<td>113</td>
<td>92</td>
<td>123</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>60</td>
<td>43</td>
<td>10</td>
<td>0</td>
<td>46</td>
<td>2</td>
<td>142</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>29</td>
<td>38</td>
<td>47</td>
<td>38</td>
<td>0</td>
<td>14</td>
<td>2</td>
<td>46</td>
</tr>
</tbody>
</table>

The primal solution found in step 1 of the primal procedure (\( a^{2}_{21} = 1 \); \( r^{3}_{23} = 1 \)) is not admissible, so the procedure opens more services. In period 1 the procedure calculates \( D \) and \( C \) equal to 35 and 15, respectively. The procedure has to choose between opening service 1 or 3. The first choice has a fixed cost equal to 100. The second has a fixed cost equal to 20. These fixed costs are transformed: \( h^1_i = \frac{100}{13} \left[ \frac{20}{13} \right] = 15.38 \); \( h^3_i = \frac{20}{19} \left[ \frac{20}{19} \right] = 2.11 \). The procedure opens service 3 in period 1 (\( a^{3}_{31} = 1 \)). The solution is still not admissible, so the procedure opens service 1 (\( a^{1}_{11} = 1 \)). It then solves the corresponding transportation problem, which has the following optimal solution:
At the end of period one, services 1 and 2 has no capacity left, service 3 has maximum capacity equal to 12.

At the beginning of period 2, there is no capacity available (because service 3 is not operating). The procedure calculates the minimum costs of (re) opening services 1 to 3 that are, respectively, equal to: 113, 48 and 9 and considers:

\[
\begin{align*}
    h_1^2 &= \frac{113}{13} \cdot \frac{35}{13} = 26.1; \\
    h_2^3 &= \frac{9}{12} \cdot \frac{35}{12} = 2.3; \\
    p_2^2 &= \frac{48}{15} \cdot \frac{35}{15} = 9.6.
\end{align*}
\]

The procedure chooses to open service three by considering \(a_{31}^1 = 0\) and \(a_{31}^2 = 1\). This is not an admissible solution (\(C = 12\) and \(D = 35\)), so the procedure needs to consider more services (re) opened. It first calculates \(p_3^2 = \frac{5}{19} \cdot \frac{23}{19} = 0.53\) and considers: \(a_{31}^2 = 0; a_{31}^1 = 1; r_{32}^2 = 1\). The solution is still not admissible, so it changes the following variables: \(a_{21}^2 = 0; a_{21}^1 = 1; r_{22}^2 = 1\).

The resolution of the corresponding transportation problem finds the following optimal solution:

\[
\begin{align*}
    h_1^3 &= \frac{114}{13} \cdot \frac{34}{13} = 26.3; \\
    p_2^3 &= +\infty; \\
    h_3^3 &= 0.
\end{align*}
\]

In period 3 it is also necessary to change the primal solution: \(h_1^3 = \frac{114}{13} \cdot \frac{34}{13} = 26.3\); \(p_2^3 = +\infty\); \(h_3^3 = 0\).

The procedure begins by considering \(r_{32}^2 = 0\) and \(r_{32}^3 = 1\) and finally it changes \(r_{32}^3\) to zero and variables \(r_{32}^2\) and \(r_{33}^3\) to 1.
The corresponding transportation problem has an optimal solution equal to:

\[
\begin{array}{cccccc}
\text{i} & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 & \\
2 & 5 & 2 & 3 & 5 & \\
3 & 9 & 10 & & & \\
\end{array}
\]

An admissible solution has been found. The primal procedure calculates the optimal transportation problem for all periods and calculates the primal objective function value equal to 701.03.

The first time the dual solution is changed is when the dual ascent procedure for variables \( \lambda_i^t \) is executed. Consider variable \( \lambda_3^1 \):

\[
\delta \leftarrow \max \left\{ \frac{1}{6} \max \{0.17 - 17\}, \frac{1}{2} \max \{0.51 - 89\}, \frac{1}{11} \max \{0.19 - 47\}, \frac{1}{5} \max \{0.12 - 14\}, \frac{1}{11} \max \{0.16 - 16\}, \frac{1}{6} \max \{0.17 - 18\}, \frac{1}{2} \max \{0.2 - 7\}, \frac{1}{12} \max \{0.20 - 50\}, \frac{1}{5} \max \{0.12 - 14\}, \frac{1}{10} \max \{0.14 - 14\}, \frac{1}{5} \max \{0.30 - 17\}, \frac{1}{2} \max \{0.8 - 8\}, \frac{1}{12} \max \{0.36 - 45\}, \frac{1}{5} \max \{0.18 - 14\}, \frac{1}{10} \max \{0.19 - 14\} \right\} = 2.6
\]

The sets \( J(t) \) are equal to \( J \), for all periods \( t \). This means that, for instance, \( \Phi(2.6,1,1) = 0 - 2.6 \times 13 = -33.8 \) and \( \Phi(2.6,1,3) = 13 + 5 + 4 - 2.6 \times 13 = -11.8 \). Slack \( SA_{i1}^1 \) would be decreased by 33.8, but slack \( SA_{i1}^3 \) would be decreased by 11.86. The procedure studies what happens with all slacks that are affected by the increase in the dual variable. As no slack will take values less than zero, the dual variable is changed. Slacks are changed accordingly:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( SA_{i1}^1 )</th>
<th>( SA_{i1}^2 )</th>
<th>( SA_{i1}^3 )</th>
<th>( SA_{i2}^2 )</th>
<th>( SA_{i2}^3 )</th>
<th>( SR_{i2}^2 )</th>
<th>( SR_{i2}^3 )</th>
<th>( SR_{i3}^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>66.20</td>
<td>120.20</td>
<td>123.20</td>
<td>96.20</td>
<td>99.20</td>
<td>123.20</td>
<td>79.20</td>
<td>80.20</td>
</tr>
</tbody>
</table>

\[ ^6 \text{Step 5 of the procedure could be changed in order to guarantee that variable } \lambda_i^t \text{ is changed only if at least one slack is increased (meaning that } \Phi(\delta, \tau, \xi) \text{ has to be greater than zero for at least one pair } (\tau, \xi) \text{ with } \tau \leq t). \text{ The procedure as it is described allows for a variable } \lambda_i^t \text{ to be increased even when that change will only decrease slacks. The same remark can be made about the dual descent procedure. The procedure as it is described changes more often the dual solution and the slacks’ values. This facilitates the construction of different solutions by the primal procedure.} \]
The dual variables \( v^j_j \) cannot be increased.

Consider also what happens with variable \( \lambda^3_3 \):

\[
\delta \leftarrow \max \left[ \frac{1}{6} \max \{0.17 - 26\}, \frac{1}{2} \max \{0.51 - 52\}, \frac{1}{11} \max \{0.19 - 28\}, \frac{1}{5} \max \{0.12 - 18\}, \frac{1}{11} \max \{0.16 - 25\}, \right.
\]
\[\frac{1}{6} \max \{0.17 - 30\}, \frac{1}{2} \max \{0.2 - 5\}, \frac{1}{12} \max \{0.20 - 29\}, \frac{1}{5} \max \{0.12 - 17\}, \frac{1}{10} \max \{0.14 - 21\},
\]
\[
\frac{1}{5} \max \{0.30 - 30\}, \frac{1}{2} \max \{0.8 - 5\}, \frac{1}{12} \max \{0.36 - 27\}, \frac{1}{5} \max \{0.18 - 18\}, \frac{1}{10} \max \{0.19 - 19\} \right] = 1.5
\]

Sets \( J_1(t) \) are equal to set \( J \) for all \( t \). This means that, for instance, \( \Phi(1.5,1,1) = 0 - 1.5 \times 19 = -28.5 \). Slack \( S \lambda^1_1 \) is equal to 20, so it would take a negative value with this increase of the dual variable \( \lambda^3_3 \). This means that \( \delta^* \) is changed to 1.5, and \( \delta \) will take a new value: 0.75. Sets \( J_1(1) \) and \( J_1(2) \) are equal to set \( J \), but \( J_1(3) = \{1,3,4,5\} \). The value of \( \Phi(0.75,1,3) \), for instance, will be calculated as \( \Phi(0.75,1,3) = 9 + 2 \times 0.75 - 19 \times 0.75 = -3.75 \) and \( \Phi(0.75,2,2) = 14.25 \). Slack \( S \lambda^2_2 \) would become less than zero, so \( \delta^* \) is changed to 0.75, and \( \delta \) will be equal to zero: the dual variable is not changed.

In this step of the heuristic, the only dual variable that is changed is \( \lambda^3_1 \).

The next procedure that changes the dual solution is the dual descent procedure for variables \( \lambda^j_i \). The only dual variable that can be decreased is \( \lambda^3_1 \):

\[
\delta \leftarrow \min \left\{ \frac{1}{2} \left( 5 - \frac{17 - 32.6}{6} \right), \frac{1}{2} \left( 6 - \frac{51 - 94.2}{2} \right), \frac{1}{5} \left( 19 - \frac{75.6}{11} \right), \frac{1}{5} \left( 12 - \frac{27}{5} \right), \frac{1}{11} \left( 16 - \frac{44.6}{11} \right), \frac{1}{11} \left( 17 - \frac{33.6}{11} \right), \right. \\
\frac{1}{12} \left( 20 - \frac{81.2}{12} \right), \frac{1}{10} \left( 12 - \frac{27}{10} \right), \frac{1}{2} \left( 14 - \frac{40}{2} \right), \frac{1}{2} \left( 8 - \frac{13.2}{2} \right), \frac{1}{10} \left( 36 - \frac{76.2}{10} \right), \frac{1}{5} \left( 18 - \frac{27}{5} \right), \frac{1}{10} \left( 19 - \frac{40}{10} \right) \right\} = 1.8
\]

Consider, for instance, slack \( S \lambda^1_1 \). The procedure calculates \( \sum_{j \in J} d^1_j = 0 \) and

\[
v^1_j - c^1_j - \sum_{j=1}^{T} d^1_j \lambda^j_i \geq 0
\]

\( \Omega(1.8,1,1) = 1.8(13 - 0) = 23.4 \). Similar calculations prove that the decrease in the dual variable is admissible, so \( \lambda^3_1 \) is changed to 0.8 and slacks are changed:
Imagine, however, that the dual descent procedure was not executed. In this case, consider the dual solution obtained after some iterations of the primal-dual heuristic:

$$\rho_2 = 54.87; \quad \rho_3 = 46; \quad \lambda_2^1 = 3.27; \quad \lambda_3^3 = 2.60$$

and variables $v_j^l$:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$SA_{i1}^1$</th>
<th>$SA_{i1}^2$</th>
<th>$SA_{i1}^3$</th>
<th>$SA_{i2}^2$</th>
<th>$SA_{i2}^3$</th>
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<th>$SR_{i2}^3$</th>
<th>$SR_{i3}^3$</th>
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<td>146.6</td>
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<td>122.6</td>
<td>146.6</td>
<td>102.6</td>
<td>103.6</td>
</tr>
</tbody>
</table>

Slacks are equal to:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$SA_{i1}^1$</th>
<th>$SA_{i1}^2$</th>
<th>$SA_{i1}^3$</th>
<th>$SA_{i2}^2$</th>
<th>$SA_{i2}^3$</th>
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<th>$SR_{i2}^3$</th>
<th>$SR_{i3}^3$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>120.2</td>
<td>123.2</td>
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<td>99.2</td>
<td>123.2</td>
<td>79.2</td>
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<tr>
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</tr>
<tr>
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<td>0</td>
<td>9</td>
<td>18</td>
<td>93</td>
<td>84</td>
<td>46</td>
<td>14</td>
<td>2</td>
</tr>
</tbody>
</table>

When the dual ascent procedure is executed for variables $\lambda_j^i$, consider variable $\lambda_3^1$:

$$\delta \leftarrow \max \left\{ \frac{1}{6} \max \{0.32.6 - 26\}, \frac{1}{2} \max \{0.65.4 - 52\}, \frac{1}{11} \max \{0.54.93 - 28\}, \frac{1}{5} \max \{0.27 - 18\}, \frac{1}{11} \max \{0.35.07 - 25\} \right\} = 6.7$$

Set $J_1(1)$ is equal to set $J$ and

$$\Phi(6.7,1,1) = 6.6 + 13.4 + 26.93 + 9 + 10.06 - 6.7 \times 19 = -61.3$$

Slack $SA_{311}^1$ is equal to zero, so this increase in the dual variable is not admissible. The procedure changes $\delta'$ to 6.7, and $\delta$ will be equal to 2.45. Set $J_1(1)$ is equal to $\{1,3,4,5\}$ and

$$\Phi(2.45,1,1) = 6.6 + 26.93 + 9 + 10.06 + 2 \times 2.45 - 2.45 \times 19 = 10.98$$

Doing the same study for all slacks, the procedure verifies that all slacks will remain greater than zero, so the dual variable is increased, and slacks become:
The dual ascent algorithm for variables $v_j^i$ will be able to increase these dual variables, improving the dual objective function value.

<table>
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<tr>
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<th>$SA^{2}_{i1}$</th>
<th>$SA^{3}_{i1}$</th>
<th>$SA^{2}_{i2}$</th>
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<td>84</td>
<td>46</td>
<td>14</td>
<td>2</td>
<td>46</td>
</tr>
</tbody>
</table>

6 Final comments and Future Work Directions

This work was motivated by the good results obtained with the computational tests performed with the primal-dual heuristic developed for DLPOCR (Dias et al., 2002).

The computational tests already performed with the heuristics, presented in this research report, indicate that the primal solutions found by the heuristics are of good quality, but the lower bounds given by the best dual solution calculated are, in general, far from the optimal solution objective function value. They also indicate that the quality of the primal solutions calculated by the C3-DLPOCR heuristic is worse than the quality of the solutions calculated for the other two problems. Systematic computational tests need to be performed in order to assess the quality of the heuristics described.

After the execution of the primal-dual heuristics it is possible to improve the best primal solution found through a local search procedure. In the DLPOCR, a local search procedure initiated with the best primal solution found by the heuristic improved significantly the best primal objective function value calculated.

One way of trying to improve the performance of these heuristics is to modify the dual solution initialization step. Instead of executing this step as described, it is possible to solve a linear programming problem and to use its optimal solution as an initial dual solution for the heuristic.

The quality of the lower bounds obtained with the primal-dual heuristic for the PLDOCR motivates the use of the lagrangean relaxation, with the subgradient optimization method, to
solve the three capacitated problems addressed in this research report. For any of the three problems, relaxing the capacity constraints in a lagrangean way results in a PLDOCR. Solving this problem heuristically instead of optimally in each iteration of the subgradient method (using the dual objective function value as a lower bound), and using primal procedures similar to the ones here presented to calculate primal admissible solutions, is capable of improving both upper and lower bounds.

There is also the possibility of formulating the three problems described considering full assignment variables instead of partial assignment variables. In this case, all problems will become much harder to solve.

7 References


