An exact method for the bi-criteria \(\{0,1\}\)-knapsack problem based on functions specialization

Carlos Gomes da Silva\(^{(2,3,a)}\), José Figueira\(^{(1,3)}\), João Clímaco\(^{(1,3)}\)

(1) Faculdade de Economia da Universidade de Coimbra
Av. Dias da Silva, 165, 3004-512 Coimbra, Portugal
Phone: +351 239 790 500, Fax: +351 239 790 514
E-mail: figueira@fe.uc.pt

(2) Escola Superior de Tecnologia e Gestão de Leiria
Morro do Lena, Alto Vieiro, 2401-951 Leiria, Portugal
Phone: +351 244 820 300, Fax: +351 244 820 301
E-mail: cgsilva@estg.iplei.pt

(3) INESC-Coimbra
Rua Antero de Quental, 199
3000-033 Coimbra, Portugal
Phone: +351 239 851 040, Fax: +351 239 824 692
E-mail: jclimaco@inescc.pt

Abstract

This paper presents a new exact method for the bi-criteria \(\{0,1\}\)-knapsack problem by using a family of weighted sum functions. These functions are used independently to obtain subsets of efficient solutions with similar structure and to build convex regions (bands) in the criteria space. These regions contain the location where it is possible to find non dominated solutions. The set of efficient solutions is determined when all the bands had been completely explored. The division of the criteria space into bands has the advantage of enabling the use of a single function which in an exploration process can make tight the gap between upper and lower bounds, which led to a more efficient exploration.

Key-words: Bi-criteria knapsack, exact methods, combinatorial optimization

\(^{a}\)Corresponding author
1 Introduction

The \{0,1\}-knapsack problem is about selecting a set of items such that the sum of their values is maximized and the sum of their weights does not exceed the knapsack capacity. This problem received a great interest over the last years, and it is extensively studied in the excellent monographs by Martello and Toth (1990) and Kellerer et al. (2003).

This paper concerns the bi-criteria case, i.e., when an additional criterion has to be taken into account to select the items. In this case, the problem does not have, in general, a single optimal solution, but a set of interesting solutions, called efficient/non dominated ones. The bi-criteria \{0,1\}-knapsack problem can be formulated as follows:

\[
\begin{align*}
\max z_1(x_1, \ldots, x_j, \ldots, x_n) &= \sum_{j=1}^{n} c_{1j} x_j \\
\max z_2(x_1, \ldots, x_j, \ldots, x_n) &= \sum_{j=1}^{n} c_{2j} x_j \\
\text{subject to:} & \\
\sum_{j=1}^{n} w_j x_j &\leq W \\
x_j &\in \{0, 1\}, j = 1, \ldots, n
\end{align*}
\]  

(1)

where \(c_{ij}\) represents the value of item \(j\) on criterion \(i\), \(i = 1, 2\), \(x_j = 1\) if item \(j\) \((j = 1, \ldots, n)\) is included in the knapsack and \(x_j = 0\) otherwise, \(w_j\) means the weight of item \(j\), and \(W\) is the overall knapsack capacity. It is assumed that \(c_{1j}, c_{2j}, W\) and \(w_j\) are positive integers and that \(w_j \leq W\) with \(\sum_{j=1}^{n} w_j > W\).

Constraints \(\sum_{j=1}^{n} w_j x_j \leq W\) (\(wx \leq W\), for short, where \(w\) is the vector with the weights and \(x\) is the vector of the decision variables) and \(x_j \in \{0, 1\}, j = 1, \ldots, n\), define the feasible region in the decision space, and their image when using the criteria functions \(z_1\) and \(z_2\) define the feasible region in the criteria space. A feasible solution, \(x\), is said to be efficient if and only if there is no feasible solution, \(y\), such that \(z_i(x) \leq z_i(y)\), \(i = 1, 2\) and \(z_i(x) < z_i(y)\) for at least one \(i\). The image of an efficient solution in the criteria space is called a non dominated solution.

The methods devoted to the generation of the set of efficient/non dominated solutions can be classified as exact or approximate ones. A method is exact when the obtained set of solutions is indeed the entire set of efficient/non dominated solutions of the problem. A method is said approximate when the nature of the solutions is not known, i.e., when it is not proved if the solution are efficient/non dominated. There are several exact methods for the bi-criteria \{0,1\}-knapsack problem: Ulungu and Teghem (1997), Visée et al. (1998), Klamroth and Wiecek (2000) and Captivo et al. (2003). The first two methods are based on branch-and-bound techniques and the remaining on dynamic programming.

Exact methods can only deal with medium size instances. Thus, another interesting line of research is the development of approximate methods: Czyzak and Jaskiewicz (1998), Ben Abdelaziz et al. (1999) and Gandibleux and Fréville (2000). They deal with the problem of inaccuracy of the obtained set of solutions. More refined approaches consists of incorporating certain degrees of exactness in the search, avoiding thus pure random processes. This kind of
research can be recognized in the works by Gandibleux et al. (2001) and Gomes da Silva et al. (2004a, 2004b and 2004c).

This paper concerns the development of a new exact method, which is based on the branch-and-bound technique. The incorporation of several criteria brings singular aspects that substantially increases the difficulty of the exploration of the tree, comparing with the single criterion case, namely because the fathom of a node is more difficult since the comparison of lower and upper bounds is less tight. The efficiency of the methods based on this technique rely on the intelligent exploration of the binary tree, which is crucial in multiple criteria problems. The effectiveness of the search can be enhanced by the knowledge of the general structure of the efficient solutions.

Let us remark that for the single criterion 0,1,-knapsack problem Balas and Zemel (1980) observed that the optimal solution of the integer problem is very similar to the optimal solution from the continuous problem (linear relaxation). This similarity led to the core concept. For instances with a large number of variables the size of the core is relatively small and increases very slowly with the number of variables.

The existence of small cores is very important for the development of efficient algorithms due to the following points (Balas and Zemel, 1980):

1. It avoids the complete sort of the items, required for obtaining better lower and upper bounds, which requires a significant part of the total computational time.

2. The problem related with the core may be used to derive better lower bounds, which enables fixing a larger number of variables.

For the bi-criteria 0,1,-knapsack problem Gomes da Silva et al. (2004c) showed empirically, by considering a large number of solutions from different instances with different sizes, that efficient solutions can be clustered in homogeneous groups. Solutions from each group have a similar structure comparing with the one that optimizes a given weighted-sum function of the criteria over the linear relaxation of the 0,1,-knapsack problem.

For the bi-criteria case these concepts can also be applied (Gomes da Silva et al., 2004c):

- A family of weighted sum functions was defined, \( \mathcal{S} = \{ p^1(x), p^2(x), ..., p^q(x) \} \), which covers the Pareto frontier of problem (1), where \( p^k(x) = \mu^k z_1(x) + (1 - \mu^k) z_2(x), k = 1, ..., q, 0 < \mu^k < 1 \).

- Taking into account the functions from \( \mathcal{S} \), the core and the core-problem were generalized to the domains of the bi-criteria problems.

- The computational experiments about the bi-criteria core size revealed the existence of the same characteristics observed in single criterion problems, namely the existence of small cores which grow slowly with the size of the problem.

- It was verified that the distribution of the core size is very “compact”, which means that the majority of the efficient solutions can be obtained by solving small core-problems according to \( \mathcal{S} \). In this way, the functions from \( \mathcal{S} \) can be used to define cores and core-problems, giving rise to a new process for generating efficient solutions from (1).
In this paper an exact method is proposed for detecting and exploring homogeneous groups, ensuring the exhaustiveness of the exploration of the decision space. The paper is structured as follows. Section 2 presents the principle of functions specialization, Section 3 is devoted to the structure of the new exact method, Section 4 presents an illustrative example, and Section 5 concerns the main conclusions and future research.

2 About functions specialization

Let \( X^{\text{eff}} \) be the set of efficient solutions of (1), defined by the union of \( q' \) subsets of efficient solutions, \( X^{\text{eff}}_k, k = 1, \ldots, q' \) (where \( q' \leq q = \text{number of functions in 3} \)):

\[
X^{\text{eff}} = \bigcup_{k=1}^{q'} X^{\text{eff}}_k
\]  

(2)

where,

\[
X^{\text{eff}}_k = \{ x^{k_1}, x^{k_2}, \ldots, x^{k_t} \}
\]

(3)

with \( x^t (t = k_1, \ldots, k_t) \) such that:

\[
\forall x^t \in X^{\text{eff}}_k : |C^{k,t}| \leq |C^{i,t}|, \forall i \in \{1, \ldots, q\}
\]

(4)

where, \( C^{k,t} \) is the bi-criteria core for solution \( x^t \) according to the use of function \( p^k(x) \).

From the point of view of a procedure based on the resolution of core-problems, the function \( p^k(x) \) is the most appropriate in order to generate solutions of \( X^{\text{eff}}_k \), because it is associated to the smaller core among the possible ones. This means that, when using an enumerative procedure, obtaining these solutions requires a smaller number of branching operations when compared with the ones required by using a different function.

These observations lead to a function specialization concept, defined as follows:

**Definition 1 (Functions specialization)** The function \( p^k(x) \in \mathcal{S} \) is said to be specialized in getting the solutions \( X^{\text{eff}}_k \) if the following condition holds:

\[
\forall x^t \in X^{\text{eff}}_k : |C^{k,t}| \leq |C^{i,t}|, \forall i \in \{1, \ldots, q\}.
\]

Thus the main idea is to use particular functions from \( \mathcal{S} \) in order to obtain specific sets of efficient solutions, building a specialization (correspondence) between functions from \( \mathcal{S} \) and subsets of efficient solutions.
The new exact method is based on the concept of functions specialization, that suggests the iterative use of functions from $\mathcal{F}$ in order to obtain the set $X_{eff}$. When using a function from $\mathcal{F}$ to solve a core-problem, we obtain a set of solutions, that experimental results revealed to be very good in the sense that a high percentage of solutions are efficient solutions from (1) (see, Gomes da Silva et al., 2004c). As it was observed, these solutions are located in precisely defined regions in the criteria space. Nevertheless, it is not guarantee that the obtained solutions:

- are efficient solutions of (1);
- correspond to the entire set of efficient solutions in the limited region.

Thus, solving the core-problems is only a “good” strategy to get good starting solutions. It is required an integrated process that proves the complete exploration of the decision space.

Let us start by observing that with the obtained solutions it is possible to define a lower frontier (see Figure 1) for the existence of efficient solutions. This frontier is placed very near to the upper frontier (see also Figure 1), built from the resolution of the linear relaxation of (1). In Figure 1, $R_{nd}$ is the region in the criteria space where it is possible to find new non dominated solutions: the region below $R_{nd}$ only contains dominated solutions and the region above $R_{nd}$ infeasible ones.

Finding the entire set of efficient/non dominated solutions of (1) consists of assuring that the region $R_{nd}$ is completely explored. This process is done by using certain functions from $\mathcal{F}$, as suggested by the concept of functions specialization. Functions from $\mathcal{F}$ are used to define bands $B(k)$ that completely cover $R_{nd}$ and, thus, the non-convex region is searched by exploring small convex regions. In Figure 2 it is shown a possible exploration, once $R_{nd}$ is contained in the region defined by the union of all the bands.
The division of $R^m$ into bands has the advantage of enabling the use of a single function from $\mathcal{F}$, that in an exploration process can make tight the gap between upper and lower bounds, which led to a more efficient exploration.

In this way, determining $X^{eff}$ consists of filtering the efficient solutions from the different bands:

$$X^{eff} = \text{Eff} \left( \bigcup_{k=1}^{p} B(k) \right)$$

where $\text{Eff} (\bullet)$ means the efficient solutions of set $\bullet$.

The exact method is based on an iterative use of functions from $\mathcal{F}$ (some may not be used). Each function is applied to:

1 - Define a bi-criteria core and solve the correspondent bi-criteria core-problem;
2 - Define a band in the criteria space;
3 - Explore the band.

These procedures will be described in the next sections. When changing a function, the consequence is the change of the organization of the branch-and-bound tree, in case of performing the exploration in the same tree, or the existence of a different tree. The organization of the trees is thus dynamic in the sequence of the iterations. The concept of function specialization gives a particular strategy to perform the exploration of the trees, and it is about the change of how the variables should be analyzed.
3.1 Defining bi-criteria cores

After choosing a function from $\mathcal{S}$, \( p^k(x) = \mu_k z_1(x) + (1 - \mu_k) z_2(x) \), the corresponding core, \( C^k = \{ i_1^k, \ldots, i_n^k \} \), is defined according to a specified heuristic, defined below. Due to the similarities between the single and the bi-criteria cores, one can use the rule by Martello and Toth (1990), i.e., build a core with approximately \( 2^{\sqrt{\alpha}} \) items around the break item, \( f_k \), of problem

\[
\max \left\{ p^k(x) : \sum_{j=1}^{n} w_j x_j \leq W, x \in [0, 1]^n \right\}.
\]

Thus, \( C^k = \{ i_1^k, \ldots, i_n^k \} = \{ f_k, f_k, \ldots, f_k + \delta \} \).

Nevertheless, it was observed (Gomes da Silva et al., 2004c) that for small size instances, the core is greater due to the inclusion of items with small weights, which do not have average efficiencies. Due to this fact, we define a core with \( 2^{\sqrt{\alpha}} \) items around the break item plus \( \alpha \) items from the left and the right of the core, with the lowest weights. The definition of a core \( C^k \) has the objective of reducing the size of the original problem, building a problem just with the variables pertaining to \( C^k \).

All the other variables are fixed to 0 and 1, according to their efficiency concerning \( p^k(x) \). This fixing process consists of assigning the value 1 to all the variables placed on the left of the core and the value 0 to all the variables on the right of the core.

As the variables not belonging to \( C^k \) are fixed at zero or one, the criteria values and the residual capacity are the following:

\[
\tilde{z}_1 = \sum_{j \in \{1, \ldots, n\} \setminus C^k} c_j^1 x_j;
\]
\[
\tilde{z}_2 = \sum_{j \in \{1, \ldots, n\} \setminus C^k} c_j^2 x_j;
\]
\[
\overline{w} = W - \sum_{j \in \{1, \ldots, n\} \setminus C^k} w_j x_j.
\]

Thus we want to solve the following bi-criteria problem:

\[
\begin{align*}
\max z_1 &= \sum_{j \in C^k} c_j^1 x_j \\
\max z_2 &= \sum_{j \in C^k} c_j^2 x_j \\
\text{subject to:} & \sum_{j \in C^k} w_j x_j \leq \overline{w} \\
& x_j \in \{0, 1\}, j \in C^k
\end{align*}
\]

In order to solve (6) we start by including the items from the core in a sequential order, until it is not possible to include any more. The variables that cannot take the value 1, will be set at 0. The corresponding solution is inserted in \( X^{eff} \).

The backtracking in the binary tree is performed by finding the last variable belonging to the core and fixed it at 1. This variable is then set to 0 and the criteria values and the residual capacity are updated accordingly. The process stops when it is not possible to find any variable fixed at 1. Otherwise, another branching is performed. Let \( i_t^k \) be the index of the variable set to 0, the upper bounds for the criteria, \( \overline{z}_1 \) and \( \overline{z}_2 \), are computed in the following way:
\[ z_1 = z_1 + \frac{c_{j_1}}{w_{j_1}} \] \hspace{1cm} (7)

\[ z_2 = z_2 + \frac{c_{j_2}}{w_{j_2}} \] \hspace{1cm} (8)

where,

- \( j_i \ (i = 1, 2) \) is the index of the item that can be inserted and has the greatest efficiency ratio for criteria \( z_i(x) \):
  \[ j_i = \arg \max_{\ell = i_{k+1}, \ldots, i_{k'}} \left\{ \frac{c_{\ell}}{w_{\ell}} : \frac{w_{\ell}}{} \right\} \]

- \( z_i \ (i = 1, 2) \) is the current value of criterion \( z_i(x) \).

**Remark 1** The upper bounds are computed separately for each function. There are several ways to define the upper bounds (see, Martello and Toth, 1990). In general, to get tight upper bounds it is required a large amount of computational time. Thus, it is necessary to assume a compromise in their selection. The expressions (7) and (8) consists of the simplest bounds, got by filling up the available residual capacity with the most efficient item (upper bound \( U_0 \), in Martello and Toth, 1990).

If the vector \((z_1, z_2)\) is dominated by the image of the solutions from \(X^{\text{eff}}\), then the backtracking operation is also performed. If the residual capacity is not enough to include the lightest item from the core, i.e., if \( \overline{w} < \min \left\{ w_j : j = i_{k+1}, \ldots, i_{k'-1} \right\} \), then the backtracking step is performed. If the item in the sequence can be inserted in the knapsack, the corresponding variable is fixed at 1, updating then the criteria values and the residual capacity. Otherwise, the variable is fixed at 0. The backtracking step is then repeated.

**Remark 2** In this procedure a node is fathomed if at least one of the following situations occurs:

1. the upper bounds for the objective function are dominated by the image of a solution from \(X^{\text{eff}}\);
2. it is not possible to include any more items;
3. the node is terminal.
The branch-and-bound procedure that is used to solve the residual problem is summarized below.

**Procedure Explore - Core \((p^k(x))\)**

Begin

Build \(C^k = \{i^k_1, ..., i^k_{n'}\}\) from \(p^k(x)\);
Set the value of \(x_j, j \in \{1, ..., n\}\) \(\setminus C^k\) accordingly with the corresponding efficiency;

\[
\begin{align*}
\bar{z}_1 & \leftarrow \sum_{j \in \{1, ..., n\} \setminus C^k} c^1_j x_j; \\
\bar{z}_2 & \leftarrow \sum_{j \in \{1, ..., n\} \setminus C^k} c^2_j x_j; \\
\bar{w} & \leftarrow W - \sum_{j \in \{1, ..., n\} \setminus C^k} w_j x_j;
\end{align*}
\]

Terminate \(\leftarrow false;\)
\(t \leftarrow 1;\)

Repeat

If \((\bar{w} - w_t \geq 0)\) Then \(x^k_{i^k_t} \leftarrow 1, \bar{w} \leftarrow \bar{w} - w^k_{i^k_t}, \bar{z}_1 \leftarrow \bar{z}_1 + c^1_{i^k_t}, \bar{z}_2 \leftarrow \bar{z}_2 + c^2_{i^k_t}\)
else \(x^k_{i^k_t} \leftarrow 0;\)

If \((\bar{w} < \min \{w_j : j = i^k_{n + 1}, ..., i^k_{n'}\})\) Then \(x^k_{i^k_t} \leftarrow 0, j = t + 1, ..., n', \text{Terminate} \leftarrow true;\)
\(t \leftarrow t + 1;\)

Until \((\text{Terminate} = true)\)
\(\mathcal{X}^{eff} \leftarrow \mathcal{X}^{eff} \cup \{x\};\)

Repeat

\[
\begin{align*}
t & \leftarrow \max \{j : x_j = 1, j = i^k_1, ..., i^k_{n - 1}\}; \\
\text{If} \ (t \text{ does not exist}) \text{ Then} \ \text{Terminate} \leftarrow true \\
\text{else} \\
& \text{Begin} \\
& \quad x_t \leftarrow 0, \bar{w} \leftarrow \bar{w} + w_t, \bar{z}_1 \leftarrow \bar{z}_1 - c^1_t, \bar{z}_2 \leftarrow \bar{z}_2 - c^2_t; \\
& \quad \text{Fill-knapsack}(\bar{z}_1, \bar{z}_2, \bar{w}, t + 1); \\
& \text{End}
\end{align*}
\]

Until \((\text{Terminate} = true)\)

End
where the procedure *Fill-knapsack* has the following specification:

\[
\textbf{Procedure Fill-knapsack}(z_1, z_2, \overline{w}, t) \\
\text{Begin} \\
\quad \text{Terminate} \leftarrow \text{false}; \\
\quad \text{Repeat} \\
\quad \quad j_1 \leftarrow \arg \max_{\ell=i_{k-1}^{\ell}, \ldots, i_{n_{k'}}^{\ell}} \left( \frac{c_{1\ell}}{w_{\ell}} : \overline{w} \geq w_{\ell} \right); \\
\quad \quad j_2 \leftarrow \arg \max_{t=i_{k-1}^{\ell}, \ldots, i_{n_{k'}}^{\ell}} \left( \frac{c_{2t}}{w_{t}} : \overline{w} \geq w_{t} \right); \\
\quad \quad z_1 \leftarrow z_1 + \left[ \frac{c_{1j_1}}{w_{j_1}} \overline{w} \right], \quad z_2 \leftarrow z_2 + \left[ \frac{c_{2j_2}}{w_{j_2}} \overline{w} \right]; \\
\quad \quad \text{If} \left( (\overline{z}_1, \overline{z}_2) \Delta Z(X^{\text{eff}}) \right) \text{ Then } \text{Terminate} \leftarrow \text{true} \\
\quad \quad \text{else} \\
\quad \quad \quad \text{Begin} \\
\quad \quad \quad \quad \text{If } (\overline{w} \geq w_{t_{k'}}) \text{ then } \overline{w} \leftarrow \overline{w} - w_{t_{k'}}; \\
\quad \quad \quad \quad \quad z_{1k} \leftarrow z_{1k} + c_{1\ell}^k \overline{w}, \quad z_{2k} \leftarrow z_{2k} + c_{2\ell}^k \overline{w}; \\
\quad \quad \quad \quad \quad \text{Else } x_{t_{k'}} \leftarrow 0; \\
\quad \quad \quad \quad \quad \text{If } (\overline{w} < \min \{w_j : j = i_{k+1}^{\ell}, \ldots, i_{n_{k'}}^{\ell} \}) \text{ then} \\
\quad \quad \quad \quad \quad \quad x_{t_{k'}} \leftarrow 0, j = t+1, \ldots, i_{n_{k'}}^{\ell}, \text{Terminate} \leftarrow \text{true}, \\
\quad \quad \quad \quad \quad \quad \text{Update } X^{\text{eff}} \text{ with } x; \\
\quad \quad \quad \quad \quad \text{If } (t = i_{n_{k'}}^{\ell}) \text{ Then } \text{Terminate} \leftarrow \text{true}, \text{Update } X^{\text{eff}} \text{ with } x; \\
\quad \quad \quad \quad \quad \text{Else } t \leftarrow t + 1; \\
\quad \quad \quad \quad \text{End} \{\text{else}\} \\
\quad \quad \quad \text{End} \{\text{else}\} \\
\quad \text{Until } (\text{Terminate} = \text{true}) \\
\text{End}
\]

3.2 Defining a band in the criteria space

There are several ways to cover \( R^{\text{nd}} \) with bands \( B(k) \). The different ways mean different gaps for the functions from \( \mathbb{E} \). Let the gap of function \( p^k(x), \Delta p^k(x) \), be the difference between the maximum and the minimum values of the function:

\[
\Delta p^k(x) = p^k_{\text{max}} - p^k_{\text{min}}, \quad k \in \{1, \ldots, q\} \tag{9}
\]

where,

- \( p^k_{\text{max}} \) is the maximum value of \( p^k(x) \) in \( R^{\text{nd}} \) and is given by the expression:

\[
p^k_{\text{max}} = \sum_{j=1}^{f_k-1} \left( \mu^k c_j^k + \left( 1 - \mu^k \right) c_j^2 \right) + \left( \mu^k c_{f_k}^k + \left( 1 - \mu^k \right) c_{f_k}^2 \right) \overline{w}^k \overline{w}_{f_k} \tag{10}
\]

- assuming that the items are ordered according to non increasing values of the efficiency ratio considering \( p^k(x) \). As the expressions of \( f_k \) and \( \overline{w}^k \) are:
\[
f_k = \min \left\{ t : \sum_{j=1}^{t-1} w_j \leq W < \sum_{j=1}^{t} w_j \right\},
\]
\[
\overline{w}^k = W - \sum_{j=1}^{t-1} w_j,
\]

\(\mu^k\) is the weight used to define \(p^k (x)\).

- \(p^k_{\text{min}}\) is the minimum value of \(p^k (x)\), defined as:

\[
R^{nd} \subset \bigcup_{k=1}^{q} \left\{ (z_1 (x), z_2 (x)) : p^k (x) \geq p^k_{\text{min}} \right\}
\]  

(11)

A measure that seems to be interesting is the percentual gap between the maximum and the minimum values of function \(p^k (x)\), which represents the relative effort of the function:

\[
\delta_k = \frac{p^k_{\text{max}} - p^k_{\text{min}}}{p^k_{\text{max}}} , \quad k \in \{1, ..., q\}
\]  

(12)

Once defined the value of \(\delta_k\), the minimum value for \(p^k (x)\), \(p^k_{\text{min}}\), is equal to \(p^k_{\text{max}} (1 - \delta_k)\).

In order to accelerate the fathom of the nodes, the values of \(\delta_k\) should be small, but, at the same time large enough to fulfill condition (11).

The definition of \(\delta_k\), or \(\Delta p^k (x)\), is not an easy task. A criterion which can be used consists of the minimization of the sum of the percentual gaps \(\delta_k\), imposing a maximum relative effort on \(p^k (x)\), \(\delta^\max_k\):

\[
\min \sum_{k=1}^{q} \delta_k 
\]

subject to:

\[
R^{nd} \subset \bigcup_{k=1}^{p} \left\{ (z_1 (x), z_2 (x)) : p^k (x) \geq p^k_{\text{max}} (1 - \delta_k) \right\}
\]

(13)

\[
0 \leq \delta_k \leq \frac{\Delta p^k}{p^k_{\text{max}}} , \quad k = 1, ..., q
\]

\[
\sum_{j=1}^{n} w_j x_j \leq W
\]

\[
x \in [0, 1]^n
\]

In the definition of the value of \(\delta^\max_k\) it is worthwhile to have into account as a reference the percentual gap of \(p^k (x)\) when \(p^k_{\text{min}}\) is obtained from \(p^k_{\text{max}}\) by rounding down its value:

\[
\delta^\pi_k = \left( \mu^k c^1_{f_k} + (1 - \mu^k) c^2_{f_k} \right) \frac{\overline{w}^k}{w_{f_k}} , \quad k \in \{1, ..., q\}
\]  

(14)

The value of \(\delta^\pi_k\) can be defined as a rule of thomb to define the value of \(\delta^\max_k\).
The branch-and-bound technique is used to explore the regions limited from above by the upper frontier and from below by \(p^k_{\min}\). As for the exploration new solutions can be obtained, reducing \(R^{nd}\). It is useful to do that exploration sequentially.

As it can be observed in Figure 2, there is overlapping of regions. A factor that enhances this overlapping is the large amount of analyzed functions, which means that it can be more interesting to define wider bands but in a smaller number than analyzing a larger number of small bands. The assessment of this trade-off is not easy to make.

In order to identify the functions to be analyzed we use the relative effort of functions from \(\mathcal{X}\). Let,

\[
\mathcal{X}_k \left( X^{eff} \right) = \left\{(z_1^0, z_2^0), (z_1^1, z_2^1), (z_2^2, z_2^2), \ldots, (z_1^{t'}, z_2^{t'+1})\right\}
\]

be the set of the images of solutions from \(X^{eff}\), \(\mathcal{X} X^{eff}\) contains all the exterior vertices of \(R^{nd}\).

### 3.2.1 Building the first band

The first band is build by considering the first function from \(\mathcal{X}\) and taking into account the value of \(\mu r_1\):

Let us define the first band as follows:

\[
B (1) = \left\{(z_1 (x), z_2 (x)) \in R^{nd} : \mu_1 (x) \geq \frac{1}{\mu_{\max}} (1 - \delta_1^{\max}), x \in X\right\}
\]

with \(\delta_1^{\max} \geq \delta_1^r\).

### 3.2.2 Building the next bands

Let us suppose that bands \(B (1), \ldots, B (k - 1)\) were already explored and that \((z_1^{t-1}, z_2^{t-1})\) from \(\mathcal{X}'(X^{eff})\) is the the first point outside \(B (k - 1)\). The function used to define the next band, \(B (k)\), is the last one in the sequence of functions of \(\mathcal{X}\), after the one which was used to define \(B (k - 1)\), such that \((z_1^{t-1}, z_2^{t-1})\) belongs to \(B (k)\) without exceeding the maximum established gap for that function. Thus, the function that was used to define \(B (k)\) is \(p^k_{\max} (x)\), where

\[
k_* = \max \left\{t : \frac{p_{\max}^t - \mu_1 t z_2^{t-1} + (1 - \mu_1 t) z_2^{t-1}}{p_{\max}^t} \leq \delta_1^{\max}, k - 1 \leq t \leq q\right\}
\]

Remark 3 As a consequence of the way how the bands are built, some functions may not be used.
Generically,

\[ B(k) = \left\{ (z_1(x), z_2(x)) \in \mathbb{R}^{2d} : p^{k_*}(x) \geq p_{\text{min}}^{k_*}, x \in X \right\} \]

where, \( p_{\text{min}}^{k_*} = p_{\text{max}}^{k_*} \left( 1 - c_{k_*}^{\text{max}} \right) \).

Let us remark that the definition is done progressively as the bands are being explored. Consequently, \( \mathcal{S}'(X_{\text{eff}}) \) is dynamic. When a band containing the last point from \( \mathcal{S}'(X_{\text{eff}}) \) is explored then it is not necessary to define additional bands.

**Remark 4** In order to keep simple the exploration of the bands, the first and the last functions from \( \mathcal{S} \) are modified. We consider \( \mu^1 = 0 \) and \( \mu^d = 1 \). In this way, it is not necessary to compute \( z_1^0 \), neither \( z_t^{d+1} \) in the sequence (16). Thus, \( z_1^0 = z_t^{d+1} = 0 \).

**Remark 5** When a band is explored, the maximum value of the function is also updated consequently, becoming the minimum value that it takes in the band.

### 3.3 Exploring a Band

Once a band \( B(k) \) is defined it is necessary to assure that it is completely explored.

**Remark 6** Note that it is impossible to avoid overlapping of the functions and consequently when one of them is analyzed, part of it was already explored.

The region of \( B(k) \) that is not yet completely explored can be expressed as follows:

\[ \widehat{B}(k) = \left\{ (z_1(x), z_2(x)) \in \mathbb{R}^{2d} : p^{k_*}(x) \geq p_{\text{min}}^{k_*}, p^t(x) < p_{\text{min}}^t, t = 1, \ldots, (k - 1)_s, x \in X \right\} \]

It is obvious that some of the previous constraints \( (p^t(x) < p_{\text{min}}^t, t = 1, \ldots, (k - 1)_s) \) may not be relevant to define \( \widehat{B}(k) \). This information must be taken into account in order to improve the exploration process. Let \( A(k) \) be the set of the indices of the constraints relevant to define \( \widehat{B}(k) \) excluding constraint \( k \), i.e., \( A(k) \subseteq \{1, 2, \ldots, (k - 1)_s\} \).

In order to keep simple the presentation of the procedure, these constraints are only implicitly considered in the original model. In this sense they are used just to verify if a solution is feasible or not.

As it was referred, the exploration of a band is done by using the branch-and-bound technique. Thus, firstly let us investigate if it is possible to reduce the number of variables to build a smaller residual problem.

#### 3.3.1 Reducing the Number of the Variables

When a lower bound for the function \( p^{k_*}(x) \left( p^{k_*}(x) \geq p_{\text{min}}^{k_*} \right) \), it is interesting to check if the value of some variables can be fixed.

There are several procedures to reduce the number of variables (see, for instance, Martello and Toth, 1990 and Kellerer et al., 2003). We use the procedure proposed by Dembo and
Hammer (1980), once it is easy to use. According to this procedure, for all variables previous to \( f_k \), the upper bound for \( p^k (x) \), \( U_j^0 \), is computed by fixing temporarily the value of the variable to 0. If this bound is smaller than a minimum value already knew, \( P_{\min}^k \), then the variable can be set definitively to 1. In a similar way, for all the variables after \( f_k \), the upper bound of \( p^k (x) \), \( U_j^1 \), is computed by setting temporarily the value of it to 1. In case of this bound is smaller than \( P_{\min}^k \), then the variable can be set definitively to 0. The computation of this limit is done in the following way:

\[
U_j^0 = p^k_j - c_j + \left( \frac{w_j^k}{w_j^k} \right) \frac{c_j}{w_j^k} < p_{\min}^k \Rightarrow x_j = 1 \quad \text{(test for } j < f_k) \]
\[
U_j^1 = p^k_j + c_j + \left( \frac{w_j^k}{w_j^k} \right) \frac{c_j}{w_j^k} < p_{\min}^k \Rightarrow x_j = 0 \quad \text{(test for } j > f_k) \]  

(21)

where,

\[
f_k = \min \left\{ h : \sum_{j=1}^{h} w_j > W \right\};
\]

\[
p^k = \sum_{j=1}^{f_k-1} c_j e^w = W - \sum_{j=1}^{f_k-1} w_j.
\]

**Example 1**

Let us consider the following instance with 10 items.

\[
\max z_1 = 91x_1 + 36x_2 + 8x_3 + 77x_4 + 93x_5 + 13x_6 + 41x_7 + 81x_8 + 11x_9 + 93x_{10}
\]

\[
\max z_2 = 35x_1 + 49x_2 + 86x_3 + 19x_4 + 50x_5 + 90x_6 + 21x_7 + 59x_8 + 63x_9 + 55x_{10}
\]

subject to:

\[
33x_1 + 45x_2 + 23x_3 + 19x_4 + 70x_5 + 100x_6 + 63x_7 + 70x_8 + 55x_9 + 10x_{10} \leq 244
\]

\[
x_j \in \{0, 1\}, j = 1, \ldots, 10
\]

Let us suppose that \( p^k (x) = 0, 5z_1 (x) + 0, 5z_2 (x) \) and that a band is defined with the following constraints:

1) capacity constraint:

\[
33x_1 + 45x_2 + 23x_3 + 19x_4 + 70x_5 + 100x_6 + 63x_7 + 70x_8 + 55x_9 + 10x_{10} \leq 244
\]

2) lower bound to the objective function in the current band:

\[
0, 5z_1 (x) + 0, 5z_2 (x) \geq 300 = p_{\min}^k
\]

3) constraints related with previous bands:

\[
0, 12z_1 (x) + 0, 88z_2 (x) \leq 308
\]

\[
0, 3z_1 (x) + 0, 7z_2 (x) \leq 336.
\]

In order to apply test (21) it is only required to take into account the knapsack constraint.

Inserting the items according to the non-increasing efficiency ratios \( x_{10} \succ x_4 \succ x_3 \succ x_1 \succ x_5 \succ x_8 \succ x_2 \succ x_9 \succ x_6 \succ x_7 \), we obtain
\[ x^* = \left(1, \frac{19}{45}, 1, 1, 0, 0, 1, 0, 1\right) \] with \( p^{k*} (x^*) = 391, 444, p_{k*} = 360, w_{k*} = 19 \) and where the break item is \( f_{k_0} = 2 \).

The result of the application of the test is summarized in Table 1.

<table>
<thead>
<tr>
<th>( x^* )</th>
<th>( x_{10} )</th>
<th>( x_4 )</th>
<th>( x_3 )</th>
<th>( x_1 )</th>
<th>( x_5 )</th>
<th>( x_8 )</th>
<th>( x_9 )</th>
<th>( x_6 )</th>
<th>( x_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U^0 )</td>
<td>326, 88</td>
<td>361, 38</td>
<td>366, 17</td>
<td>359, 61</td>
<td>386, 06</td>
<td>387, 56</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( U^1 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>376, 5</td>
<td>348, 5</td>
<td>362, 94</td>
</tr>
</tbody>
</table>

**Decision** F NF NF F NF N F NF NF F NF

F-Fix, NF-Do not fix

Table 1: Reduction procedure considering only the knapsack constraint

With the procedure (21) it was possible to set the value of three variables.

If the constraints \( p^t (x) < p^{t\min} \), \( \forall t \in A(k) \) were considered in the reduction step, then it is possible that a higher number of variables are fixed at their optimal values. The problem to be considered is thus:

\[
\max \left\{ p^{k*} (x) : x \in X, p^t (x) < p^{t\min}, \forall t \in A(k) \right\}
\] (22)

Using the following result, adapted from Nemhauser and Wolsey (1988) to the previous problem, where \( \overline{c} \) means the reduced price of variable \( x_j \) in the optimal solution of the relaxed problem (22), \( x^* \), and \( \widehat{x} \) is a feasible solution:

**Proposition 1** If the variable \( x_j \) is not a basic variable and is at its lower (upper) bound in the optimal solution, \( x^* \), of the relaxed problem of (22) and \( p^{k*} (x^*) + \overline{c}_j \leq p^{k*} (\widehat{x}) \) \( (p^{k*} (x^*) - \overline{c}_j \leq p^{k*} (\widehat{x})) \), then there exists an optimal solution for the integer problem with \( x_j \) in the lower (upper) bound.

In summary, we have that:

a) If \( x_j^* = 0 \) and \( p^{k*} (x^*) + \overline{c}_j \leq p^{k*} (\widehat{x}) \), then \( x_j \) can be definitely fixed at 0;

b) If \( x_j^* = 1 \) and \( p^{k*} (x^*) - \overline{c}_j \leq p^{k*} (\widehat{x}) \), then \( x_j \) can be definitely fixed at 1.

Note that this proposition is equivalent to (21) when there is only one constraint.

The application of Proposition 1 requires the resolution of the relaxed problem of (22). But, it worths to verify if the constraints related to \( A(k) \) are relevant or, at least, if their number can be reduced.

The answer to the first question is given by the following result.

Let us consider that:

\[ x^+ = \arg \max \left\{ p^{k*} (x) : x \in \text{Conv} (X) \right\}, \] and

\[ \Psi = \left\{ j \in A(k) : p^j (x^+) \geq p^{\min}\right\}. \]
Proposition 2 If \( \Psi \neq \emptyset \) then \( \max \{ p^k (x) : x \in \text{Conv} (X) , p^l (x) < p^l_{\min} , \forall t \in A (k) \} < p^k (x^+) \). Otherwise, \( \max \{ p^k (x) : x \in \text{Conv} (X) , p^l (x) < p^l_{\min} , \forall t \in A (k) \} = p^k (x^+) \).

Proof. If \( \Psi \neq \emptyset \) then \( x^+ \notin \text{Conv} (X) \cap \{ x : p^l (x) < p^l_{\min} , \forall t \in A (k) \} \) but \( x^+ \in \text{Conv} (X) \cap \{ x : p^l (x) < p^l_{\min} , \forall t \in A (k) \} \). Thus,

\[
\max \{ p^k (x) : x \in \text{Conv} (X) , p^l (x) < p^l_{\min} , \forall t \in A (k) \} < \max \{ p^k (x) : x \in \text{Conv} (X) , p^l (x) < p^l_{\min} , \forall t \in A (k) \setminus \{ j \in \Psi \} \} = p^k (x^+) .
\]

If \( \Psi = \emptyset \), \( x^+ \in \text{Conv} (X) ) , p^l (x) < p^l_{\min} , \forall t \in A (k) \) \( . \) Then, \( \max \{ p^k (x) : x \in \text{Conv} (X) , p^l (x) < p^l_{\min} , \forall t \in A (k) \setminus \{ j \in \Psi \} \} = p^k (x^+) \).

By the other side, \( \{ x : x \in \text{Conv} (X) ) , p^l (x) < p^l_{\min} , \forall t \in A (k) \} \subset \{ x : x \in \text{Conv} (X) \} \) and then, \( \max \{ p^k (x) : x \in \text{Conv} (X) , p^l (x) < p^l_{\min} , \forall t \in A (k) \} = p^k (x) \).

Thus, if \( \Psi = \emptyset \), the constraints of \( A (k) \) are not necessary and we can apply the procedure (21). In case of \( \Psi \neq \emptyset \), the optimal solution of \( \max \{ p^k (x) : x \in \text{Conv} (X) , p^l (x) \leq p^l_{\min} , \forall t \in A (k) \} \) verifies the following conditions:

\[
\begin{align*}
\text{ux}^* &= W \\
\exists t &\in \Psi \text{ such that } p^l (x) = p^l_{\min}
\end{align*}
\]

Let \( \Psi' \) be the set of constraints of \( \Psi \) that are binding and \( \rho \) a member of \( \Psi' (\rho \in \Psi') \). It comes that:

\[
\max \{ p^k (x) : x \in \text{Conv} (X) , p^l (x) < p^l_{\min} , \forall t \in A (k) \} = \\
\max \{ p^k (x) : x \in \text{Conv} (X) , p^l (x) = p^l_{\min} \}
\]

(24)

This equality means that only one constraint from \( A (k) \) is relevant for finding the optimal solution of \( \max \{ p^k (x) : x \in \text{Conv} (X) , p^l (x) \leq p^l_{\min} , \forall t \in A (k) \} \).

Let us now show how a binding constraint is identified, i.e., how to find a constraint such that \( p^l (x) = p^l_{\min} , t \in \Psi \).

One possibility consists of solving the linear relaxed problem:

\[
\max \left\{ p^k (x) : \sum_{j=1}^{n} w_j x_j \leq x , p^l (x) \leq p^l_{\min} , x \in [0,1]^n \right\} , \text{with } \rho \in \Psi
\]

(25)

Let \( x^" \) be an optimal solution from (25). If this solution is feasible for all the constraints from \( \Psi \), then the constraint is identified. Otherwise, another \( \rho \in \Psi \) is selected and once again the problem (25) is solved. This process is repeated until a solution that verifies all the constraints from \( \Psi \) is achieved. When this happens, we use the reduced prices of the variables and the Proposition 1 is applied.

Example 2

Let \( x^* = \left( 1 , \frac{19}{45}, 1,1,1,0,0,1,0,1 \right) \) be the optimal solution found in Example 1. This solution verifies the constraints which define the band. Indeed,
\[ 0, 12z_1(x^*) + 0, 88z_2(x^*) = 320, 68 \geq 308 \\
0, 3z_1(x^*) + 0, 7z_2(x^*) = 345, 7 \geq 336. \]

Now let us verify if it is advantageous to take them into account. We start by maximizing \( p^k(x) \)
subject to the constraints \( \sum_{j=1}^n w_j x_j \leq W \) and \( 0, 12z_1(x) + 0, 88z_2(x) \leq 308. \) The optimal solution of
this problem is \( x' = (1; 0, 79; 0, 27; 1; 0; 0; 1; 0; 1). \)

This solution does not verify the constraint \( 0, 3z_1(x') + 0, 7z_2(x') \cdot 336 \):

The optimal solution is presented in Table 2. The relevant constraint is now identified.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_6 )</th>
<th>( x_7 )</th>
<th>( x_8 )</th>
<th>( x_9 )</th>
<th>( x_{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x' )</td>
<td>1</td>
<td>–</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( F )</td>
<td>19,86</td>
<td>–</td>
<td>–</td>
<td>18,96</td>
<td>10,03</td>
<td>–21,65</td>
<td>–5,40</td>
<td>6,8</td>
<td>–10,01</td>
<td>28,52</td>
</tr>
<tr>
<td>Decision</td>
<td>F</td>
<td>–</td>
<td>–</td>
<td>F</td>
<td>NF</td>
<td>F</td>
<td>NF</td>
<td>NF</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

F-Fix, NF-Do not fix

Table 2: Reduction process using the constraints corresponding to \( A(k) \)

On the one side, applying the Proposition 1, when a variable with value 1 has a reduced price larger
than \( 373, 07 - 360 = 13, 07 \) this variable can be fixed at 1. On the other side, if a variable has value 0
and with a reduced price lower than \(-13, 07\), it can be fixed a 0. We conclude that it is possible to fix
the value of four variables. One more than when taking only into account the knapsack constraint.

3.3.2 Solving the residual problem

After fixing the value of some variables, let \( F_k \) be the set of the indices of the free variables
(not fixed), \( N_0^k \) the set of the indices of the variables fixed at 0, and \( N_1^k \) the set of indices of the
variables set to 1. The set \( F_k \) will be used to define the residual problem. Let \( F_k = \{ i_1^k, \ldots, i_{m'}^k \} \).

This problem is solved by adapting the procedure \( \text{Explore} - \text{Core} \). Although, the following
modifications are required:
1. In order to assure that the obtained solutions belong to the band that is being explored, a variable will not be fixed at 1 if it results in a solution outside the band. In this case, the variable is set to 0.

2. In order to take into account the minimum value of the function $p^k \times (x), p^k_{\min}$, a backtrack step in the exploration of the binary tree is performed if an upper bound for $p^k \times (x)$ is smaller than $p^k_{\min}$.

The upper bound is given by the following expression:

$$
\overline{p^k} = p^k + \left[ \frac{c_j}{w_{j_3}} \right]
$$

where $j_3 = \arg \max_{\ell = i_k^1, \ldots, i_k^n} \left\{ \frac{c_\ell}{w_\ell} : w_\ell \geq w_{j_3} \right\}$.

We can now present the procedure used to explore a band is given below.

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Explore - Band ($p^k \times (x), p^k_{\min}, A(k), X_{\text{eff}}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Begin</td>
<td>Reduce the number of variable and define $N_0^k, N_1^k$ and $F^k$;</td>
</tr>
<tr>
<td></td>
<td>$x_j \leftarrow 1, \forall j \in N_1^k; x_j \leftarrow 0, \forall j \in N_0^k$;</td>
</tr>
<tr>
<td></td>
<td>$z_1 \leftarrow \sum_{j \in N_1^k} c_1^j, z_2 \leftarrow \sum_{j \in N_1^k} c_2^j, p^k \leftarrow \sum_{j \in N_1^k} c_j$;</td>
</tr>
<tr>
<td></td>
<td>$\overline{w} \leftarrow W - \sum_{j \in N_1^k} w_j$;</td>
</tr>
<tr>
<td></td>
<td>Terminate$\leftarrow$ false;</td>
</tr>
<tr>
<td></td>
<td>$t \leftarrow 1$;</td>
</tr>
<tr>
<td></td>
<td>Fill-KnapsackBand($z_1, z_2, p^k, \overline{w}, t$);</td>
</tr>
<tr>
<td>Repeat</td>
<td>$t \leftarrow \max { j : x_j = 1, j = i_k^1, \ldots, i_{k-1}^n }$;</td>
</tr>
<tr>
<td></td>
<td>If (t does not exist) Then Terminate$\leftarrow$ true</td>
</tr>
<tr>
<td></td>
<td>else Begin</td>
</tr>
<tr>
<td></td>
<td>$x_t \leftarrow 0, \overline{w} \leftarrow \overline{w} + w_t, z_1 \leftarrow z_1 - c_1^t, z_2 \leftarrow z_2 - c_2^t, p^k \leftarrow p^k + c_i^t$;</td>
</tr>
<tr>
<td></td>
<td>Fill-KnapsackBand($z_1, z_2, p^k, \overline{w}, t + 1$);</td>
</tr>
<tr>
<td></td>
<td>End {else}</td>
</tr>
<tr>
<td>Until</td>
<td>(Terminate = true)</td>
</tr>
<tr>
<td>End</td>
<td></td>
</tr>
</tbody>
</table>
The procedure \textit{Fill-KnapsackBand} as the following specification:

\textbf{Procedure} \textit{Fill-KnapsackBand}($z_1, z_2, p^k, \overline{w}, t$)

\textbf{Begin}

\textit{Terminate} $\leftarrow$ false;

\textbf{Repeat}

\hspace{1em} $j_1 \leftarrow \arg \max_{\ell = i_k^1, \ldots, i_n^k} \left\{ \frac{c_{1\ell}}{w_\ell} : \overline{w} \geq w_\ell \right\}$;

\hspace{1em} $j_2 \leftarrow \arg \max_{\ell = i_k^2, \ldots, i_n^k} \left\{ \frac{c_{2\ell}}{w_\ell} : \overline{w} \geq w_\ell \right\}$;

\hspace{1em} $j_3 \leftarrow \arg \max_{\ell = i_k^3, \ldots, i_n^k} \left\{ \frac{c_{3\ell}}{w_\ell} : \overline{w} \geq w_\ell \right\}$;

\hspace{1em} $\overline{z}_1 \leftarrow z_1 + \overline{c_{1j_1}}; \overline{z}_2 \leftarrow z_2 + \overline{c_{2j_2}}; \overline{p}^k = p^k + \overline{c_{3j_3}};$

\hspace{1em} If $\left( \overline{p}^k < p_{\text{min}}^k \right)$ and $(\overline{z}_1, \overline{z}_2)$ is dominated by $Z \left( X_{\text{eff}}^k \right)$ Then

\hspace{1.5em} $\overline{x}_j \leftarrow x_j, \forall j = 1, \ldots, t, \text{Temp} \leftarrow \text{Temp} \cup \text{CodDec} (\overline{x})$;

\hspace{1em} If $(\overline{z}_1, \overline{z}_2)$ is dominated by $Z \left( X_{\text{eff}}^k \right)$ or $\overline{p}^k < p_{\text{min}}^k$ Then

\hspace{1.5em} \textit{Termination} $\leftarrow$ true

\hspace{1em} Else

\hspace{1.5em} \textbf{Begin}

\hspace{2em} If $(\overline{w} - w_i \geq 0)$ and \( \mu^a \left( \overline{z}_1 + c_{1i}^k \right) + (1 - \mu^a) \left( \overline{z}_2 + c_{2i}^k \right) > p_{\text{min}}^a, \forall a \in A (k) \) Then

\hspace{2.5em} \textbf{Begin}

\hspace{3em} $x_{i_k^j} \leftarrow 1, \overline{w} \leftarrow \overline{w} - w_{i_k^j}, \overline{p}^k \leftarrow \overline{p}^k + c_{i_k^j};$

\hspace{3em} $\overline{z}_1 \leftarrow \overline{z}_1 + c_{1i_k^j}^k; \overline{z}_2 \leftarrow \overline{z}_2 + c_{2i_k^j}^k;$

\hspace{2.5em} \textbf{End} \{if\}

\hspace{2em} Else

\hspace{2.5em} $x_{i_k^j} \leftarrow 0;$

\hspace{2em} If $(\overline{w} < \min \{ w_j : j = i_k^1, \ldots, i_n^k \})$ Then

\hspace{2em} \textbf{Begin}

\hspace{3em} $x_{j_k^k} \leftarrow 0, j = t + 1, \ldots, i_k^k;$

\hspace{3em} \textit{Termination} $\leftarrow$ true;

\hspace{3em} Update $X_{\text{eff}}$ with $x$;

\hspace{2em} \textbf{End} \{if\}

\hspace{2em} If \( (t = i_k^k) \) Then \textit{Termination} $\leftarrow$ true,

\hspace{2em} Update $X_{\text{eff}}$ with $x$;

\hspace{2em} Else \( t \leftarrow t + 1; \)

\hspace{2em} \textbf{End} \{else\}

\hspace{1em} \textbf{Until} (\textit{Termination} = true)

\textbf{End}

In this procedure CodDec ($\overline{x}$) means the coding $\overline{x}$ of in a decimal system.
Remark 7 Let us remark that as mentioned in (3.2) the same function can be used several times which means an enlargement of the previously explored band. In order to avoid starting from the beginning of the entire procedure, a list with the coding of the fathomed nodes by the upper bound is built.

Remark 8 At the end nodes it is possible to obtain solutions that do not belong to the current band and are not dominated by any of the available solutions. This is due to the fact that the upper bounds were not tight enough to assure a premature fathom.

3.4 Overall presentation of the exact method

The exact method starts by finding the family of weighted sum functions, $\mathcal{P}$. For each function it is defined a bi-criteria core and the corresponding core-problem is solved. The obtained solutions are used to define a lower frontier for the existence of non-dominated solutions. A new iteration cycle is repeated, which consists of defining and exploring bands until all the regions where it is possible to get non-dominated solutions are completely explored. In this cycle, the lower frontier is updated, which can reduce the number of bands to be explored.

The comprehensive presentation of the method is given below.

**Method ME1-KnpUni**

Begin

Compute $\mathcal{P} = \{p^1(x), \ldots, p^q(x)\}$

$k \leftarrow 1$

$X^{eff} \leftarrow \emptyset$

Repeat

Explore $- Core\left(p^k(x)\right)$

Consider $X^{eff}_k$ the set of the obtained solutions

Update $X^{eff}$ with $X^{eff}_k$

$k \leftarrow k + 1$

Until ($k = q$)

$k \leftarrow 1$

Temp $\leftarrow \emptyset$

Build $\kappa'_k(X^{eff})$

Repeat

Find $k^*$

Define $p_{min}^{k^*}$

If ($p^{k^*}(x) \neq p^{(k-1)^*}(x)$) Then Temp $\leftarrow \emptyset$

Build $A(k)$

Explore $- Band \left(p^{k^*}(x), p_{min}^{k^*}(x), A(k), X^{eff}\right)$

Consider $X^{eff}$ the updated set of solutions

Build $\kappa'_k(X^{eff})$

Until ($\kappa'_k(X^{eff}) \subset \bigcup_{t=1}^{k} B(t)$)

End
The method ME1-KnpUni enables the determination of all the efficient solutions of the problem (1) once every efficient solution $x$ has the image in the region $R_{nd}$ which belongs to $\bigcup_{k} B(k)$. Since all the bands were completely explored using the branch-and-bound technique were the fathom condition reveal the impossibility of obtaining new efficient solutions with images in the band, then $x$ was obtained in the exploration of a given band.

### 4 An illustrative example

Let us consider the following instance:

\[
\begin{align*}
\text{max } z_1 &= 91x_1 + 36x_2 + 8x_3 + 77x_4 + 93x_5 + 13x_6 + 41x_7 + 81x_8 + 11x_9 + 93x_{10} \\
\text{max } z_2 &= 35x_1 + 49x_2 + 86x_3 + 19x_4 + 50x_5 + 90x_6 + 21x_7 + 59x_8 + 63x_9 + 55x_{10} \\
\text{subject to:} & \\
33x_1 + 45x_2 + 23x_3 + 19x_4 + 70x_5 + 100x_6 + 63x_7 + 70x_8 + 55x_9 + 10x_{10} &\leq 244 \\
x_j &\in \{0, 1\}, j=1, \ldots, 10
\end{align*}
\]

We will use

\[X = \left\{ x \in \{0, 1\}^{10} : 33x_1 + 45x_2 + 23x_3 + 19x_4 + 70x_5 + 100x_6 + 63x_7 + 70x_8 + 55x_9 + 10x_{10} \leq 244 \right\} \]

The extreme points from the linear relaxation are: $(323, 67; 360, 1), (384, 2714; 356, 7286), (394, 8; 353, 4), (444, 4571; 334, 4286), (458, 2; 324, 6889), (468, 6; 263, 27333)$.

The family of functions $\mathcal{S}$ is presented in Table 3.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\mu^k$</th>
<th>$p^k(x) = \mu^k z_1(x) + (1 - \mu^k) z_2(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0,263505</td>
<td>0,263505z_1(x) + 0,9736496z_2(x)</td>
</tr>
<tr>
<td>2</td>
<td>0,1464535</td>
<td>0,1464535z_1(x) + 0,8535465z_2(x)</td>
</tr>
<tr>
<td>3</td>
<td>0,2583212</td>
<td>0,2583212z_1(x) + 0,7416788z_2(x)</td>
</tr>
<tr>
<td>4</td>
<td>0,3455995</td>
<td>0,3455995z_1(x) + 0,6544005z_2(x)</td>
</tr>
<tr>
<td>5</td>
<td>0,6345069</td>
<td>0,6345069z_1(x) + 0,3654931z_2(x)</td>
</tr>
<tr>
<td>6</td>
<td>0,9727126</td>
<td>0,9727126z_1(x) + 0,0728745z_2(x)</td>
</tr>
</tbody>
</table>

Table 3: Family of functions $\mathcal{S}$

4.1 Defining cores

Once the problem has only a very small number of variables, we set the core size as three elements plus an additional one.

Sorting all the items according to their efficiency ratios taking into account the functions from $\mathcal{S}$, we get the results summarized in Table 4.

4.2 Exploring cores

After defining the cores and settled the corresponding partitions $(C^k, H^k, L^k)$ it was verified that there are only four distinct core problems which should be solved. They correspond to
The functions $p^1(x)$, $p^2(x)$, $p^5(x)$ and $p^6(x)$. The partitions $(C^3, H^3, L^3)$ and $(C^4, H^4, L^4)$ are equal to the partitions already obtained.

In order to illustrate the process of solving the core problem, we consider the function $p^2(x) = 0.1464535z_1(x) + 0.8535465z_2(x)$. For this function, the core is composed of the items $C^2 = \{10, 9, 8, 5\}$. Thus, the core problem to be solved is:

$$\begin{align*}
\max z_1 &= 212 + 93x_5 + 81x_8 + 11x_9 + 93x_{10} \\
\max z_2 &= 189 + 50x_5 + 59x_8 + 63x_9 + 55x_{10} \\
\text{subject to:} & \\
124 + 70x_5 + 70x_8 + 55x_9 + 10x_{10} &\leq 244 \\
x_j &\in \{0, 1\}, \; j = 5, 8, 9, 10
\end{align*}$$

The corresponding branch-and-bound tree is shown in Figure 3 and the information concerning each node of the tree is given in Table 5.

Thus, for example,

- in the node $N^1$, the fathom by the residual capacity occurred because $\overline{w} = 59 < \min \{w_8, w_9\}$. 

<table>
<thead>
<tr>
<th>Function $p^k(x)$</th>
<th>$C^k$</th>
<th>$H^k$</th>
<th>$L^k$</th>
<th>$\overline{z}_1$</th>
<th>$\overline{z}_2$</th>
<th>$\overline{w}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^1(x)$</td>
<td>${4, 6, 8} \cup {10}$</td>
<td>${1, 2, 3, 9}$</td>
<td>${5, 7}$</td>
<td>146</td>
<td>233</td>
<td>88</td>
</tr>
<tr>
<td>$p^2(x)$</td>
<td>${5, 8, 9} \cup {10}$</td>
<td>${1, 2, 3, 4}$</td>
<td>${6, 7}$</td>
<td>212</td>
<td>189</td>
<td>124</td>
</tr>
<tr>
<td>$p^3(x)$</td>
<td>${5, 8, 9} \cup {10}$</td>
<td>${1, 2, 3, 4}$</td>
<td>${6, 7}$</td>
<td>212</td>
<td>189</td>
<td>124</td>
</tr>
<tr>
<td>$p^4(x)$</td>
<td>${2, 8, 9} \cup {10}$</td>
<td>${1, 3, 4, 5}$</td>
<td>${6, 7}$</td>
<td>269</td>
<td>1900</td>
<td>99</td>
</tr>
<tr>
<td>$p^5(x)$</td>
<td>${2, 7, 8} \cup {10}$</td>
<td>${1, 4, 5}$</td>
<td>${3, 6, 9}$</td>
<td>261</td>
<td>104</td>
<td>122</td>
</tr>
</tbody>
</table>

Table 4: Cores, starting solutions and residual capacity
<table>
<thead>
<tr>
<th>Node</th>
<th>$\bar{z}$</th>
<th>$\overline{w}$</th>
<th>$\overline{r}$</th>
<th>Fathoming reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N^0$</td>
<td>(212,189)</td>
<td>124</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$N^1$</td>
<td>(316,307)</td>
<td>59</td>
<td>-</td>
<td>Residual capacity</td>
</tr>
<tr>
<td>$N^2$</td>
<td>(305,244)</td>
<td>114</td>
<td>(456,340)</td>
<td>-</td>
</tr>
<tr>
<td>$N^3$</td>
<td>(386,303)</td>
<td>44</td>
<td>-</td>
<td>Residual capacity</td>
</tr>
<tr>
<td>$N^4$</td>
<td>(305,244)</td>
<td>114</td>
<td>(456,325)</td>
<td>-</td>
</tr>
<tr>
<td>$N^5$</td>
<td>(398,294)</td>
<td>44</td>
<td>-</td>
<td>End node</td>
</tr>
<tr>
<td>$N^6$</td>
<td>(212,189)</td>
<td>124</td>
<td>(376,331)</td>
<td>-</td>
</tr>
<tr>
<td>$N^7$</td>
<td>(223,252)</td>
<td>69</td>
<td>-</td>
<td>Residual capacity</td>
</tr>
<tr>
<td>$N^8$</td>
<td>(212,189)</td>
<td>124</td>
<td>(376,293)</td>
<td>Upper bound</td>
</tr>
</tbody>
</table>

Table 5: Information about the nodes

- in the node $N^8$, it was verified that the fathom occurred because:

  $\overline{z}_1 = 212 + \left[ \frac{124 \cdot 81}{70} \right] = 376; \overline{z}_2 = 189 + \left[ \frac{124 \cdot 59}{70} \right] = 293$ and the solution $(376, 293)$ is dominated by the solution $(386, 303)$. Thus, there is no need of additional branching.

With the exploration of all the cores associated with the functions from $\mathcal{G}$ it is obtained the set of solutions (which are shown in Figure 4):

$X^{eff} = \mathcal{E}f \left( X^{eff}_1 \cup X^{eff}_2 \cup X^{eff}_3 \cup X^{eff}_4 \cup X^{eff}_5 \cup X^{eff}_6 \right) = \{(1110000111), (1011100011), (1011100101)\}$. The images of $X^{eff}$ in the criteria space are $(320, 347), (373, 308)$ and $(443, 304)$.

Using these solutions, $\kappa'_1 (X^{eff}) = \{(0, 347), (320, 308), (373, 304), (443, 0)\}$.

### 4.3 Defining and exploring bands

Two adaptations were made: $\mu^1 = 0$ and $\mu^6 = 1$.

**Defining the first band**

Considering the first function $p^1(x) = z_2(x)$. The maximum value is obtained with the point $(323, 67; 360, 1)$ in the upper frontier. Consequently, $p_{\text{max}}^1 = 360, 1$. The best value for this function with solutions from $X^{eff}$ is 347. The reference gap is $\delta_1^* = \frac{360, 1 - 347}{360, 1} = 0, 036$. Using this value we defined heuristically $\delta_{\text{max}}^1 = 0, 05$. This gap gives rise to the constraint: $p^1(x) \geq (1 - 0, 05) \times 360, 1 = 342, 095$. Thus, the first band consists of the set (see Figure 5)

$B (1) = \{(z_1(x), z_2(x)) \in R^{nd} : p^1(x) \geq 342, 095, x \in X \}$.

**Exploring the first band**
A - Reducing the number of variables

Computing the items efficiencies by using $p^1(x)$, the following order is obtained: $x_{10} < x_3 < x_9 < x_2 < x_1 < x_4 < x_6 < x_8 < x_5 < x_7$. The variable corresponding to the break item is $x_6$ and its efficiency is $\frac{c^2_6}{w_6} = 0.9$, with $p^1 = 307$ and $w^1 = 59$. The application of the upper bound proposed by Dembo and Hammer (1980) is summarized in Table 6.

Then, three variables were fixed and thus: $F^1 = \{1, 2, 4, 5, 6, 8, 9\}$, $N^1_1 = \{3, 10\}$ e $N^1_0 = \{7\}$.

B - Solving the first residual problem

After applying the procedure Explore-Bands, it is obtained the following set of non dominated solutions:
Function $p_k(x)$ | $p_k^{\text{max}}$ | $p_k^{\text{min}}$ | Relative effort for $p_k(x)$
--- | --- | --- | ---
$p_1(x)$ | 342.095 | 338.00 | 0.01197
$p_2(x)$ | 360.76 | 335.36 | 0.07040
$p_3(x)$ | 364.09 | 333.35 | 0.08444
$p_4(x)$ | 372.45 | 331.78 | 0.10921
$p_5(x)$ | 409.40 | 326.58 | 0.20230
$p_6(x)$ | 468.60 | 320.00 | 0.31712

Table 7: Relative effort for functions in the second band

$Z(X^{\text{eff}}) = \{(293, 348), (320, 347), (332, 308), (361, 317), (373, 308), (443, 304)\}$. Using the updated set $Z(X^{\text{eff}})$, we get:

$\kappa_2'(X^{\text{eff}}) = \{(0, 348), (293, 347), (320, 338), (332, 317), (361, 308), (373, 304), (443, 0)\}$. Let us note that in the exploration of the band were obtained points that do not belong to it.

**Defining the second band**

According to the new set $Z(X^{\text{eff}})$, the first point that do not belong to the first band is (320, 308). It is used to identify the second function which led to the second band. To determine the second function it is computed the relative effort for each function (notice that the maximum value for $p_1(x)$ was updated, becoming equal to 342,095). The obtained results are presented in Table 7. Let us assume that $\delta_2^{\text{max}} = \delta_3^{\text{max}} = \delta_4^{\text{max}} = \delta_5^{\text{max}} = \delta_6^{\text{max}} = 0.1$. Thus, based on the obtained results, the next function to be used is $p_3(x)$, once the corresponding effort is lower than the maximum and for the functions $p_4(x), p_5(x)$ and $p_6(x)$ the effort is greater than the allowed maximum.

The second function to be considered is thus $p_3(x)$, with the minimum value equal to $p_3^{\text{min}} = 364.09 \times (1 - 0.1) = 327.681$ and the second band (see Figure 5), can be written as:

$$B(2) = \{(z_1(x), z_2(x)) \in R^{10} : p_1(x) \leq 342,095, p_3(x) \geq 327,681, x \in X\}.$$**Exploring the second band**

A - Reducing the number of variables

In the construction of the second band it is only used a function of the type $p^t(x) \leq p_{\text{min}}^t$. Solving the problem $\max \{p_3(x) : p_1(x) \leq 342,095, x \in \text{conv}(X)\}$ it is obtained the solution and the reduced prices presented in Table 8.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$x_9$</th>
<th>$x_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^0$</td>
<td>0.1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.37</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.32</td>
</tr>
<tr>
<td>$x_2^0$</td>
<td>19.86</td>
<td>5.25</td>
<td>41.21</td>
<td>17.04</td>
<td>0</td>
<td>-18.41</td>
<td>-27.26</td>
<td>2.99</td>
<td>0</td>
</tr>
</tbody>
</table>

**Decision**

F-Fix, NF-Do not fix
Based on these results, $F^2 = \{1, 2, 4, 5, 6, 7, 8, 9\}$, $N^2_1 = \{3, 10\}$ and $N^2_0 = \emptyset$. The decision about whether we should fix or not the value of the variables was defined taking into account the difference between the maximum and the minimum values to the function $p^3(x)$. The maximum value is $p^3(x^*) = 363,3537$ with the minimum value equal to $327,681$. Then, $p^3(x^*) - p^3_{\min} = 35,6727$.

### B - Solving the second residual problem

When solving the residual problem with the procedure Explore-Band, it is obtained a new non-dominated solution: $(388, 318)$. This solution dominates $(373,308)$ and $(361,317)$, considered in the previous iteration. The new set $Z^{eff}$ is now equal to \{(293,348),(320,347),(332,338),(388,318),(443,304)\}. Consequently, $\kappa_3^{d}(X^{eff}) = \{(0, 348),(293, 347),(320, 338),(332, 318),(388, 304),(440,0)\}$.

#### Defining the third band

According to the new set $\kappa_3^{d}(X^{eff})$, the first point that do not belong to the second band is $(332, 318)$. The effort corresponding to the functions $p^3(x), p^4(x), p^5(x)$ and $p^6(x)$ is presented in Table 9 (notice now that the maximum value of $p^3(x)$ becomes equal to $327,681$).

All the functions, excepting $p^3(x)$, exceed the maximum allowed effort. In this case, the function $p^3(x)$ remains the function to define the next band.

Keeping the function, the minimum value of $p^3(x)$ is $p^3_{\min} = 321,616$. The next band to be explored (see Figure 5) can be written as:

\[ B(3) = \{(z_1(x), z_2(x)) \in R^{nd} : p^3(x) \leq 327,681, p^3(x) \geq 321,616, x \in X\}. \]

#### Exploring the third band

### A - Reducing the number of variables
When solving the problem \( \max \{ p^3(x) : p^3(x) \leq 327,681, x \in \text{conv}(X) \} \) it is now obtained the solution and the reduced prices that are presented in Table 10.

When solving the problem \( \max \{ p^3(x) : p^3(x) \leq 327,681, x \in \text{conv}(X) \} \) it is now obtained the solution and the reduced prices that are presented in Table 10.

<table>
<thead>
<tr>
<th>Function ( p^k(x) )</th>
<th>( p^k_{\text{max}} )</th>
<th>( p^k_{\text{min}} )</th>
<th>Relative effort for ( p^k(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p^3(x) )</td>
<td>321,616</td>
<td>114,436</td>
<td>0,64418</td>
</tr>
<tr>
<td>( p^4(x) )</td>
<td>372,454</td>
<td>153,101</td>
<td>0,58894</td>
</tr>
<tr>
<td>( p^5(x) )</td>
<td>409,403</td>
<td>281,087</td>
<td>0,31342</td>
</tr>
<tr>
<td>( p^6(x) )</td>
<td>468,6</td>
<td>443</td>
<td>0,05463</td>
</tr>
</tbody>
</table>

Table 11: Relative effort for functions from \( \mathcal{F} \), fourth band

Table 10: Reducing the number of variables in the third band

Naturally, it is not possible to fix the value of any more variables. Then, \( F^3 = \{1,2,3,4,5,6,7,8,9,10\} \), \( N^3 = N^3_0 = \emptyset \).

**B - Solving the third residual problem**

Continuing the exploration of the previous binary tree (built in the exploration of the previous band), it was not possible to find new efficient solutions. The set \( Z(X^{eff}) \) is kept the same and \( \kappa'_4(X^{eff}) = \kappa'_4(X^{eff}) \).

**Defining the fourth band**

According to \( \kappa'_4(X^{eff}) \), the first point not belonging to the third band is \((443,0)\). The relative effort corresponding to the functions \( p^3(x), p^4(x), p^5(x) \) and \( p^6(x) \) is presented in Table 11 (once again the maximum value of \( p^3(x) \) is updated, being now equal to 321,616).

The only function that observes the maximum allowed effort is \( p^6(x) \).

The next band to be explored (see Figure 5) can be written as:

\[
B(4) = \{ (z_1(x), z_2(x)) \in \mathbb{R}^n : p^3(x) \leq 321,616; p^6(x) \geq 443, x \in X \}.
\]
Table 13: Efficient and non dominated solutions of the problem

<table>
<thead>
<tr>
<th>$x$</th>
<th>$z(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1011010011)</td>
<td>(293, 348)</td>
</tr>
<tr>
<td>(1110000111)</td>
<td>(320, 347)</td>
</tr>
<tr>
<td>(1110100011)</td>
<td>(332, 338)</td>
</tr>
<tr>
<td>(0111100101)</td>
<td>(388, 318)</td>
</tr>
<tr>
<td>(1011100101)</td>
<td>(443, 304)</td>
</tr>
</tbody>
</table>

*Exploring the fourth band*

A - Reducing the number of variables

The problem related with the reduction of the number of variables is now
\[
\max p^6(x) : wx \leq x, p^3(x) \leq 321, 616, x \in [0, 1]^{10},
\]
where the optimal solution and the reduced prices of the variables are presented in Table 12.

<table>
<thead>
<tr>
<th>$x^*$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$x_9$</th>
<th>$x_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>c_j</td>
<td>64.6</td>
<td>0</td>
<td>-10.4</td>
<td>61.8</td>
<td>37.0</td>
<td>-67</td>
<td>-9.4</td>
<td>25.0</td>
<td>-33</td>
<td>85</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Decision</th>
<th>F</th>
<th>NF</th>
<th>F</th>
<th>F</th>
<th>NF</th>
<th>NF</th>
<th>F</th>
<th>F</th>
</tr>
</thead>
</table>

F: Fix, NF: Do not fix

Because $p^6(x^*) = 468.6$, all the variables with value 1 and with a reduced price lower than $468.6 - 443 = 25.6$ can be fixed at 1, and all the variables with value 0 and with a reduced price lower than $-25.6$ can be fixed at 0. Six variables were fixed.

B - Solving the fourth residual problem

The resolution of the fourth residual problem does not led to new efficient solutions. The set $Z(X_{eff})$ is kept the same and because there is no point outside the fourth band, the process stops. The non dominated solutions of the problem are the ones belonging to $Z(X_{eff})$. In Table 13 these solutions are presented as well as their composition in the decision space.
Figure 5: Defining bands

$p^3(x) \geq 321,616$

$p^4(x) \geq 342,095$

$p^5(x) \geq 327,681$
5 Conclusions and future research

The new proposed method is based on a new search strategy of the criteria space. The original objective functions are replaced by weighted sum functions, used separately to generate specific sets of efficient solutions. In the \{0,1\}-knapsack problem this correspondence is particularly favorable due to the results obtained by Balas and Zemel (1980).

This paper aims at showing a new methodology for generate the exact set of efficient solutions for the \{0,1\}-knapsack problem. Nevertheless, it is important to assess the computational quality, comparing with the other available methods in the literature.

In the description made, the bands were explored independently. From a computational point of view, the search could benefit from an integrated exploration of the bands, where the information about the branching in one band can be used in the exploration of the following trees. Indeed, when changing bands only the way as the items are sorted is changed.

Acknowledgements: The authors would like to acknowledge the support from MONET research project (POCTI/GES/37707/2001, Fundação para a Ciência e Tecnologia, Portugal).
References


