

Geometrical configuration of the Pareto frontier of bi-criteria $\{0,1\}$ -knapsack problems

Carlos Gomes da Silva^(2,3), João Clímaco^(1,3), José Figueira^(1,3)

(1) Faculdade de Economia da Universidade de Coimbra

Av. Dias da Silva, 165, 3004-512 Coimbra, Portugal

Tel: +351 239 760 500, Fax: +351 239 790 514

E-mail: figueira@fe.uc.pt

(2) Escola Superior de Tecnologia e Gestão de Leiria

Morro do Lena, Alto Vieiro, 2401-951 Leiria, Portugal

Tel: +351 244 820 300, Fax: +351 244 820 301

E-mail: cgsilva@estg.iplei.pt

(3) Instituto de Engenharia de Sistemas e Computadores de Coimbra

Rua Antero de Quental, 199

3000-033 Coimbra, Portugal

Tel.: +351 239 851 040, Fax: +351 239 824 692

E-mail: jclimaco@inescc.pt

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Abstract

This paper deals with three particular models of the bi-criteria $\{0,1\}$ -knapsack problem: equal weighted items, constant sum of the criteria coefficients, and the combination of the two previous models. The configuration of the Pareto frontier is presented and studied. Several properties on the number and the composition of the efficient solutions are devised. The connectedness of the efficient solutions is investigated and observed for the entire set of efficient solutions of the third model. The models are highly structured and induce singular properties in the geometry of Pareto frontiers when compared with the ones of randomly generated instances. This aspect can increase the knowledge about the generation of efficient solutions for general bi-criteria $\{0,1\}$ -knapsack problems. The models can also be useful in generic $\{0,1\}$ -multiple criteria problems.

Key-words: Bi-criteria knapsack problems, Pareto frontier, Connectedness, Combinatorial Optimization.

1 Introduction

The bi-criteria $\{0,1\}$ -knapsack problem can be stated as follows:

$$\begin{aligned}
 \max z_1(x_1, \dots, x_j, \dots, x_n) &= \sum_{j=1}^n c_j^1 x_j \\
 \max z_2(x_1, \dots, x_j, \dots, x_n) &= \sum_{j=1}^n c_j^2 x_j \\
 \text{s.t. :} & \\
 \sum_{j=1}^n w_j x_j &\leq W \\
 x_j &\in \{0, 1\}, j = 1, \dots, n
 \end{aligned} \tag{1}$$

where c_j^i represents the value of item j on criterion $i, i = 1, 2; x_j = 1$ if item j ($j = 1, \dots, n$) is included in the knapsack and otherwise, $x_j = 0$, w_j means the weight of item j , and W is the overall knapsack capacity.

We assume that c_j^1, c_j^2, W and w_j are positive integers and that $w_j \leq W$ with $\sum_{j=1}^n w_j > W$.

Constraints $\sum_{j=1}^n w_j x_j \leq W$ ($wx \leq W$, for short, where $w = (w_1 \dots w_j \dots w_n)$ and $x^T = (x_1 \dots x_j \dots x_n)$ and $x_j \in \{0, 1\}, j = 1, \dots, n$) define the *feasible region* in the *decision space*, and their image when using the criteria functions z_1 and z_2 define the feasible region in the *criteria space*.

Definition 1 (Efficient solution) A feasible solution, x , is said to be efficient if and only if there is no feasible solution, y , such that $z_i(x) \leq z_i(y), i = 1, 2$ and $z_i(x) < z_i(y)$ for at least one i .

The image of an efficient solution in the criteria space is called a *non-dominated solution*.

The solutions that can be obtained by maximizing weighted-sums of the criteria are called *supported efficient/non-dominated* solutions. But there is a set of solutions, called *non-supported efficient/non-dominated* solutions, that cannot be obtained this way, because despite being efficient/non-dominated, they are convex dominated by weighted-sums of the criteria (Steuer, 1986). The non-supported non-dominated solutions are located in the dual gaps of consecutive supported non-dominated solutions.

For uniform and randomly generated instances a Pareto frontier (PF) similar to the one presented in Figure 1 is obtained, *i.e.*, a frontier which has the configuration of a concave function. But very distinct frontiers can be obtained when some constraints are imposed on the criteria coefficients values. The distinctive aspects appear not only in the configuration of the PF but also in the composition of the efficient solutions.

This paper is about the study of the configuration of PF and the formalization of some interesting and nice properties about efficient solutions for different cases of problem (1). The following related models are considered:

Model 1 (Equal weighted items): $w_j = k, j = 1, \dots, n;$

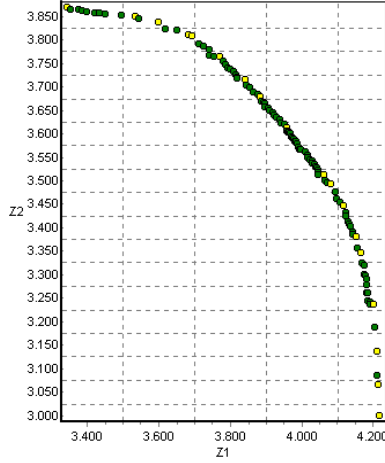


Figure 1: PF of a generic instance with $n = 100$

Model 2 (Constant sum of the criteria coefficients): $c_j^1 + c_j^2 = \alpha, j = 1, \dots, n$;

Model 3 (Constant sum of the criteria coefficients and equal weighted items): $c_j^1 + c_j^2 = \alpha$
and $w_j = k, j = 1, \dots, n$.

In the above models, the constraints on the coefficients are called restrictive conditions and it is also assumed that all the coefficients are positive integer numbers.

When studying Models 1-3, there is an important issue related with the number of variables having value 1 in a solution, *i.e.*, with the cardinality of solutions:

$$|x| = \sum_{j=1}^n x_j = x^T x \quad (2)$$

where x^T is the transpose of x .

The geometric place, in the criteria space, where solutions with the same cardinality are located is called here *iso-item line* (*plane* or *hyperplane* if the number of criteria is three or more, respectively). The definition of an iso-item line, where all solutions have cardinality δ is given below, δ , being the order of the iso-item line:

Definition 2 (Iso-item line of order δ) $ISO(\delta) = \{(z_1(x), z_2(x)) : |x| = \delta, x \in \{0, 1\}^n\}$.

Since efficient solutions have, in general, different cardinalities, the PF is composed of several mixed iso-item lines.

For the majority of multiple criteria (even bi-criteria) combinatorial optimization problems the computation of the entire set of efficient solutions is quite difficult, even impossible for large size instances. The investigation of the properties of the set of efficient solutions that can make

easier such a computation is thus an important line of research. One property, that has a key role in multiple criteria linear programming algorithms, is the *connectedness* of the set of extreme efficient solutions (Isermann, 1977; Steuer, 1986; Helbig, 1990).

In multiple criteria combinatorial optimization problems the connectedness issue was already investigated in the work by Ehrgott and Klamroth (1997). In this paper, the absence of connectedness was proved for the shortest path and minimal spanning tree problems. The authors conclude by saying that it is not known any combinatorial optimization problem where the connectedness exist for all the possible instances. The connectedness is important because it implies that all the efficient solutions can be computed by means of local search methods.

In this paper we also investigate the connectedness for all the three above referred to models. The following definition of connectedness is assumed.

Definition 3 (Connectedness) *The set of efficient solutions is said to be connected iff there is a path of adjacent efficient solutions (in sense of multiple criteria linear programming) between any pair of solutions.*

The paper is organized as follows. Section 2 presents Model 1. Section 3 is devoted to Model 2. Section 4 concerns Model 3. Section 5 presents some extensions to the general multiple criteria $\{0,1\}$ -problems. Finally, Section 6 presents the main conclusions of this work.

2 Model 1: Equal weighted items

In this model, all the weights have the same value:

$$w_j = k, k \geq 0, j = 1, \dots, n \quad (3)$$

If $k = 0$ there is no conflict between criteria: their optima are achieved simultaneously at the same solution. In this case, the PF is composed of a single solution.

A more interesting situation occurs when $k > 0$. Replacing (3) in problem (1), the knapsack constraint becomes:

$$\sum_{j=1}^n kx_j \leq W \Leftrightarrow \sum_{j=1}^n x_j \leq \frac{W}{k}.$$

Due to the discrete nature of the decision variables, the inequality above is equivalent to:

$$\sum_{j=1}^n x_j \leq \left\lfloor \frac{W}{k} \right\rfloor.$$

Letting $\left\lfloor \frac{W}{k} \right\rfloor = \ell$, the problem (1) can now be re-written as follows:

$$\begin{aligned}
\max z_1(x_1, \dots, x_j, \dots, x_n) &= \sum_{j=1}^n c_j^1 x_j \\
\max z_2(x_1, \dots, x_j, \dots, x_n) &= \sum_{j=1}^n c_j^2 x_j \\
s.t. : \\
\sum_{j=1}^n x_j &\leq \ell \\
x_j &\in \{0, 1\}, j = 1, \dots, n
\end{aligned} \tag{4}$$

Figure 2 shows the usual configuration of the PF for randomly generated instances of problem (4).

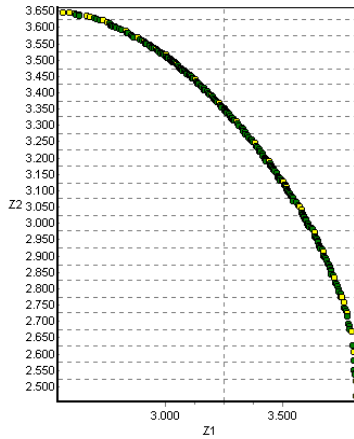


Figure 2: PF (Model 1) of an instance with $n = 100$

Comparing the configuration of the PF shown in the figure above, with the one shown in Figure 1, it can be seen that the concavity is preserved. The distinctive aspects appear in the composition of the efficient solutions.

2.1 Elementary properties

The knapsack constraint is a cardinality constraint: all the feasible solutions do not have more than ℓ variables with the value equal to 1.

For the resolution of problem (4) the cardinality constraint (inequality $\sum_{j=1}^n x_j \leq \ell$) can be strengthened to an equality $\left(\sum_{j=1}^n x_j = \ell \right)$ due to the following proposition.

Proposition 1 *If x^+ is an efficient solution of problem (4) then x^+ has exactly ℓ variables with value equal to 1: $|x^+| = \ell$.*

Proof. *Suppose that x^+ is an efficient solution such that $|x^+| < \ell$. Since $x^+ + e_j$ (e_j is a vector with n components all equal to 0 except the j component which is equal to 1) is also feasible if*

$x_j^+ = 0$, and since the criteria coefficients are positive, then it is possible to improve the value of both criteria. As a result, x^+ , cannot be an efficient solution, which leads to a contradiction. As for a solution x^+ such that $|x^+| > \ell$ is not feasible, the proof is complete. ■

From the proposition above all the efficient solutions have the same cardinality. In this case, the PF is the $ISO(\ell)$.

The particularity that all efficient solutions have the same cardinality means that when moving from one solution to another, the number of variables that change from 0 to 1 is equal to the number of variables that change from 1 to 0. Otherwise, the solution does not belong to the $ISO(\ell)$.

If x' and x'' denote two different efficient solutions of problem (4) it comes that:

$$|x' - x''| = 2\pi, \quad \pi \in \{1, \dots, \min\{\ell, n - \ell\}\} \quad (5)$$

Remark 1 Let x^t, x^{t+1} and x^{t+2} be three efficient solutions of problem (4) with consecutive images in the criteria space. It can be shown, by using examples, that we can obtain $|x^t - x^{t+2}| < |x^t - x^{t+1}|$. This is not a much desired feature when hopping to obtain a transition rule between efficient solutions based on small and systematic interchanges of items.

With all equal coefficients, the items are exclusively compared in terms of their criteria values and it is easy to build a dominance relation among them (see Figure 3 where the arc (1,2) means that item 1 dominates item 2). From this comparison, some items cannot be included without including others. By similar reasons, if some items are not included then other items cannot be included.

The proposition that follows provides a rule for fixing the value of certain variables. Let Δ denote the dominance relation, where $a\Delta b$ means that a dominates b .

Proposition 2 If x^+ is an efficient solution of problem (4) and $(c_j^1, c_j^2) \Delta (c_i^1, c_i^2), i, j \in \{1, \dots, n\}$ then $x_j^+ \geq x_i^+$.

Proof. Suppose that x^+ is efficient and that $x_j^+ = 0$ and $x_i^+ = 1$. Then, $x^+ + e_j - e_i$ is feasible, and because $(c_j^1, c_j^2) \Delta (c_i^1, c_i^2)$ it is possible to increase the value of both criteria, which contradicts the fact that x^+ is efficient. ■

An interesting application of Proposition 2 appears when it is investigated, from a given efficient solution, the change of the value of some variables in order to obtain other efficient solutions. To see this, consider the dominance relation of a subset of items shown in Figure 3. If x_1 is set to 0 then, in order to obtain another efficient solution x_2, x_3, x_4 and x_5 must be fixed to 0. On the other hand, if x_4 is set to 1 then x_1, x_2 and x_3 must be set to 1. Thus, complementing the value of some variables induces other complementarities, which can lead to infeasible solutions, since the overall dominance relation must be preserved.

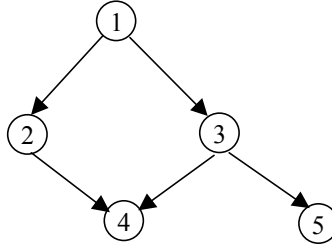


Figure 3: Dominance relation among a subset of items

Empirically, it can be expected that when some items at the bottom of the dominance relation (that are dominated by many other items in a hierarchic sense) are included in the knapsack, a great instability in the structure of the dominance will be caused in the sense that a large number of variables have to be commuted in order to preserve the dominance relation.

The dominance relation among items induces precedence relations for the variables. Consider $A \equiv \{(j, k) : (c_j^1, c_j^2) \Delta (c_k^1, c_k^2), j, k \in \{1, \dots, n\}\}$. Then, problem (4) has a strengthened formulation if the set of constraints $x_k \leq x_j$, for $(j, k) \in A$ is added to it.

2.1.1 Particular instances

Proposition 2 can enhance reduction procedures on the number of variables, once under the conditions of the proposition setting x_i to 1 implies that x_j is also set to 1. If x_j is set to 0 then, x_i can be set to 0.

The application of this proposition is particularly effective for instances of problem (4) that observe the following two additional conditions:

$$\begin{cases} a) & (c_j^1, c_j^2) \Delta (c_k^1, c_k^2) \quad \forall j \in L, k \in \{1, \dots, n\} \setminus L \\ b) & |L| \geq \ell \end{cases}$$

where $L \equiv \{j \in \{1, \dots, n\} : \nexists t \in \{1, \dots, n\} \text{ where } (c_t^1, c_t^2) \Delta (c_j^1, c_j^2)\}$.

Figure 4 shows an example of coefficients that observe condition a). The first quadrant contains the items belonging to L and the third quadrant comprises all the items dominated by those belonging to L .

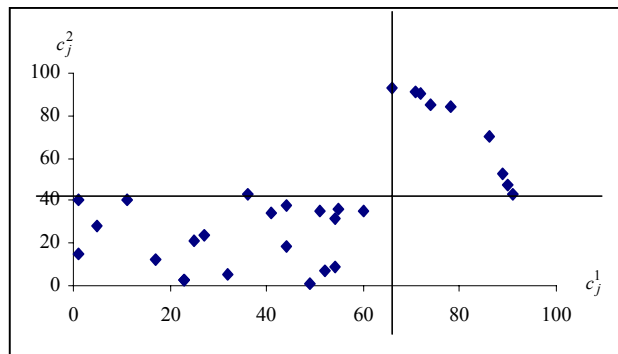


Figure 4: A particular case of problem (4)

If conditions a) and b) are fulfilled we have the following result,

Proposition 3 *If x^+ is an efficient solution of problem (4) where the criteria coefficients observe conditions a) and b), then $x_j^+ = 0, \forall j \notin L$.*

Proof. *Suppose that x^+ is an efficient solution with $x_j^+ = 1$ for a given $j \notin L$. Because $|x^+| = \ell \leq |L|$ this means that at least one variable related to L has value 0, $x_t^+ = 0, t \in L$. Due to condition a) x^+ is dominated by $x^+ - e_j + e_t$ which is also a feasible solution, i.e., x^+ cannot be an efficient solution. ■*

According to this proposition all the items belonging to the third quadrant (the ones not pertaining to L) could be put aside from further consideration. Although, it should be noticed:

Remark 2 *Despite L contains an enough number of items to build solutions with cardinality ℓ and all the criteria coefficients are non-dominated among them, it does not mean that all the combinations of ℓ items from L generate efficient solutions. This is illustrated with the following example. Consider the criteria coefficients: $(29, 77), (31, 69), (44, 56), (50, 50)$. All these vectors are non-dominated among them, but $(29, 77) + (50, 50)$ dominates $(31, 69) + (44, 56)$.*

2.2 Outline of some important aspects when computing the entire set of efficient solutions

The entire set of efficient solutions comprises supported and non-supported efficient solutions. We start by presenting important aspects about the supported and then about the non-supported ones.

An important issue which is useful for computing both types of solutions comes from the nature of the feasible decision space: all the feasible points of $\overline{X} \equiv \{x \in \{0, 1\}^n : x^T x = \ell\}$ are extreme points of $\text{Conv}(\overline{X})$ (see Figure 5 for an illustration, where $X \equiv \text{Conv}(\overline{X})$), where $\text{Conv}(\overline{X})$ is the *convex hull* of \overline{X} (Nemhauser, 1988).

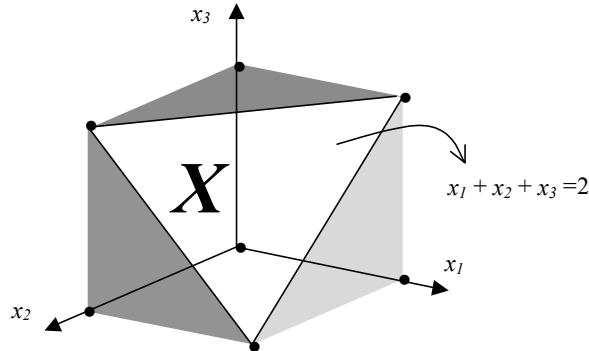


Figure 5: The cubic decision space

2.2.1 Computing the supported solutions

Let us start by the problem of computing the entire set of supported efficient solutions. For problem (4), this computation is trivial, since there exists a link between linear and integer solutions, as established below.

2.2.1.1 A link between linear and integer solutions

Since all the extreme points of $\text{Conv}(\overline{X})$ are integer, it follows:

Proposition 4 *If x^0 is a supported efficient supported solution of (4) then there exists a $\lambda \in]0, 1[$ such that x^0 is an optimal solution of the problem $\max\{\lambda z_1(x) + (1 - \lambda) z_2(x) : x^T x = \ell, x \in [0, 1]^n\}$.*

Each solution from the set of supported solutions can be obtained through the maximization of weighted-sum functions in the linear relaxation of the problem (4). It suffices to select the ℓ items with the highest value in the objective function. This is a greedy procedure, which produces the optimal solution. The proof of this trivial result can be obtained by verifying that:

1. Efficient supported solutions are obtained by optimizing weighted-sum functions, z_λ , which converts the problem into a single criterion one;
2. The problem to be optimized is an usual knapsack problem, where the efficiency of the items is equal to their coefficients, c_j , in the weighted-sum function;
3. Applying the Dantzig optimal heuristic to the continuous problem an upper bound, $\overline{z_\lambda}$, is obtained, which is equal to the lower bound $\underline{z_\lambda} = \sum_{j=1}^{\ell} c_j = \overline{z_\lambda} = \sum_{j=1}^{\ell} c_j + 0 \times \frac{c_{\ell+1}}{1}$, with $c_1 \geq c_2 \geq \dots \geq c_n$, i.e., $x_j = 1, j = 1, \dots, \ell$ and $x_j = 0, j = \ell + 1, \dots, n$.

2.2.1.2 Adapting the simplex algorithm with bounded variables

From the previous section, which establishes a link between the linear and the integer problems, the computation of the entire set of supported solutions of problem (4) can be done in a systematic way by using the simplex method with bounded variables. In Gomes da Silva *et al.* (2003, 2004) this process is defined for the general continuous bi-criteria knapsack problem, which due to the proposition above can be used to obtain efficient solutions for problem (4).

The fact that the knapsack constraint of (4) is only a cardinality one, it induces important simplifications in the process: the reduced prices in the simplex tableau are simple differences between criteria coefficients of the non-basic variables and the basic one, the update of the value of the entering and of the leaving variable consists of no more than a commutation of values.

The main features of the procedure are summarized below.

A bi-criteria simplex tableau with bounded variables contains only a current basic variable, x_f , which is the $\ell + 1$ variable with the highest value in the objective function. Of course, the value of the basic variable is 0, which means that the solution is a degenerated one. The lines (in the simplex tableau) related to the reduced prices for each criterion are simply $c_j^i - c_f^i, i = 1, 2$.

In order to avoid obtaining a solution that maintains the value of criterion z_2 but improves the value of criterion z_1 (a weakly efficient solution) the objective function above referred to is a weighted-sum function of the criteria, $\lambda z_1(x) + (1 - \lambda) z_2(x)$, with λ strictly greater than 0.

Let x^0 be a solution that maximizes the criterion z_2 . The solution x^0 remains the optimal solution of the weighted-sum function $\lambda^0 z_1(x) + (1 - \lambda^0) z_2(x)$ while

$$\lambda' \leq \lambda^0 \leq \lambda^* = \min \{ \phi_j, j = 1, \dots, n, j \neq f \} \quad (6)$$

where
 $\lambda' = 0$, and

$$\phi_j = \begin{cases} -\left(c_j^2 - c_f^2\right) & \text{if } \left(x_j = 1, \left(c_j^1 - c_f^1\right) - \left(c_j^2 - c_f^2\right) < 0\right) \\ \frac{-\left(c_j^2 - c_f^2\right)}{\left(c_j^1 - c_f^1\right) - \left(c_j^2 - c_f^2\right)} & \text{if } \left(x_j = 0, \left(c_j^1 - c_f^1\right) - \left(c_j^2 - c_f^2\right) > 0\right) \\ +\infty & \text{otherwise} \end{cases}, \quad \begin{matrix} j = 1, \dots, n, \\ j \neq f \end{matrix} \quad (7)$$

The variable corresponding to the minimum ϕ_j, x_{j^*} , is the variable which enters the basis. Generically, the efficient pivoting process for problems with the constraints of problem (4) is as follows. If $x_{j^*} = 0$, then its value as basic variable remains zero and x_f leaves the basis also with value 0. In this case, it is obtained a different basis for the same extreme point. If $x_{j^*} = 1$, then its value as basic variable becomes zero and x_f leaves the basis and takes the value 1.

Let x^1 be the solution after the pivoting process (adjacent efficient solution). The edge, in the criteria space, that links $[x^0, x^1]$ can contain other non-dominated solutions. They are obtained by entering the basis the non-basic variables such that $\lambda^* \left(c_j^1 - c_f^1\right) + (1 - \lambda^*) \left(c_j^2 - c_f^2\right) = 0$, *i.e.*, entering the basis a non-basic efficient variable. In order to obtain an organized process of computing all the alternative integer solutions of a given weighted-sum function, the procedure presented in Steuer (1986, Chapter 4) can be used. Basically, a list is created to control the basis already explored and avoid redundancy in the pivoting.

Notice that, in general, for linear multiple criteria problems, intermediary solutions are not associated with basic solutions, and consequently are not detected by checking if $\lambda^* \left(c_j^1 - c_f^1\right) + (1 - \lambda^*) \left(c_j^2 - c_f^2\right) = 0$. Indeed, all the segment that links two adjacent extreme non-dominated solutions are also non-dominated solutions of the problem. But, as problem (4) has integer variables, not all the solutions in such segment correspond to feasible solutions. Problem (4) is particular in this sense. The presented process of obtaining intermediary solutions is possible due to the fact that all these points corresponds to extreme points of the decision space and to Proposition 4.

When a non-basic efficient variable enters the basis, two outcomes are possible due to the degeneracy of the basic solution: it is obtained another basis for the same point, or, it is obtained a new efficient solution (Steuer, 1986).

The process is repeated until $\lambda^* > 1$, with x^1 taking the role of x^0 and λ' equal to λ^* from the previous iteration. When $\lambda^* > 1$, all the efficient supported solutions were determined.

Let $\Theta^{(1)}$ be the set of efficient solutions of problem (4) obtained by using the above process. For this set we have the following result which is a natural consequence of the linear multiple criteria programming:

Remark 3 *The set $\Theta^{(1)}$ is connected.*

In general, for problem (4) there is no direct correspondence between Euclidean distances of adjacent efficient solutions in the criteria and the decision spaces, as quoted in Remark 1. However, for the extreme efficient solutions the transition to the nearest solution, in the criteria space, is carried out by moving to one of the nearest solutions in the decision space. Indeed, since the integrality conditions of the problem can be relaxed and all the extreme efficient solutions can be obtained by solving the correspondent linear problem, the multiple criteria linear programming property of adjacent efficient basis holds (Steuer, 1986): adjacent efficient basis differ in a single variable. Since the solution is binary and the cardinality constraint is binding, changing the basis implies that a variable changes from 0 to 1 and another from 1 to 0, or *vice-versa*. Consequently, if x^t and x^{t+1} are two adjacent extreme efficient solutions of problem (4), then $|x^t - x^{t+1}| = 2$.

2.2.2 Computing the non-supported solutions

Although all the extreme points of $\text{Conv}(\overline{X})$ are integer points, there are some extreme points that correspond to efficient solutions which cannot be obtained with efficient pivoting from other efficient points, *i.e.*, are efficient non-supported solutions. To illustrate this, consider the following instance:

$$\begin{aligned} \max z_1 &= 6x_1 + 1x_2 \\ \max z_2 &= 1x_2 + 3x_3 \\ \text{s.t.} & : \text{Conv}(x_1 + x_2 + x_3 \leq 1, x_1, x_2, x_3 \in \{0, 1\}) \end{aligned}$$

For this problem, if a simplex tableau is associated with point B , in Figure 6 (in this figure $X \equiv \text{Conv}(x_1 + x_2 + x_3 \leq 1, x_1, x_2, x_3 \in \{0, 1\})$) the only efficient pivoting consists of moving to point A . Although, in the same tableau, it is possible to go from B to C , by entering the basis variable x_2 to replace x_1 (which is not an efficient pivoting).

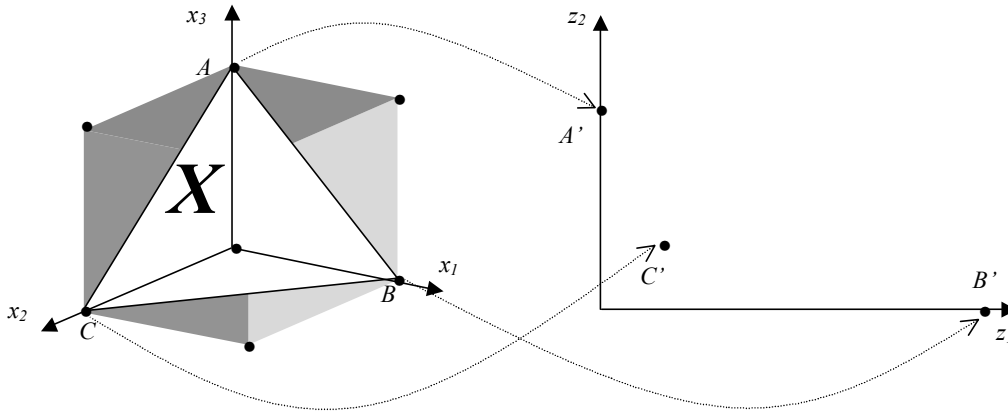


Figure 6: Extreme points in the decision space and non-dominated solutions

Of course, there can exist extreme points in the decision space that do not correspond to non-dominated solutions at all.

2.2.2.1 Using Murty's (1983) ranking method

Upon these results, using efficient pivoting do not guarantee the determination of the entire set of efficient solutions. A different process must be considered. Once all the extreme points of $Conv(\bar{X})$ are integer points and the underlying pivoting process is very simple, a ranking method can be interesting for this purpose, namely the one presented by Murty (1983, pp.158). This ranking method is based on the three very important results from the geometry of the simplex method (suppose the following problem $\max\{g(x) : x \in X\}$ where X is a convex polyhedron and $g(x)$ a linear function):

1. Let x^t and x^{t+1} be a pair of extreme points of X . Then there exists an edge path on X between x^t and x^{t+1} .
2. Let x^{t+1} be an optimum basic feasible solution and let x^t be another basic feasible solution. There exists an edge path on X from x^t to x^{t+1} with the property that as we walk along this edge path the value of the objective function is nondecreasing.
3. Suppose that x^1, \dots, x^r are the r best extreme points of X . Then the $(r+1)$ th best extreme point of X , x^{r+1} , can be considered to be an adjacent extreme point of x^1, \dots, x^r and maximizes $g(x)$ among these points.

The ranking process, applied to our problem, starts with the optimal solution of criterion $z_2(x)$ and uses this function as the role of $g(x)$. The solution that optimizes criterion $z_1(x)$ is used to obtain the minimum value for $z_2(x)$, which is useful to stop the determination of adjacent extreme points that have a lower value in this criterion. An additional filtering operation is required for retaining only the efficient solutions. This filtering is simply the dominance test among solutions. The set of filtered solutions is the efficient set of problem (4).

3 Model 2: Constant sum of the criteria coefficients

Let us assume a constant sum for the criteria coefficients:

$$c_j^1 + c_j^2 = \alpha, j = 1, \dots, n \quad (8)$$

Replacing constraint (8) in problem (1) gives:

$$\begin{aligned} \max z_1(x_1, \dots, x_j, \dots, x_n) &= \sum_{j=1}^n c_j^1 x_j \\ \max z_2(x_1, \dots, x_j, \dots, x_n) &= \sum_{j=1}^n (\alpha - c_j^1) x_j \\ s.t. : \\ \sum_{j=1}^n w_j x_j &\leq W \\ x_j &\in \{0, 1\}, j = 1, \dots, n \end{aligned} \quad (9)$$

The configuration of the PF of problem (9) consists of stretches of lines. Figure 7 shows some possible configurations.

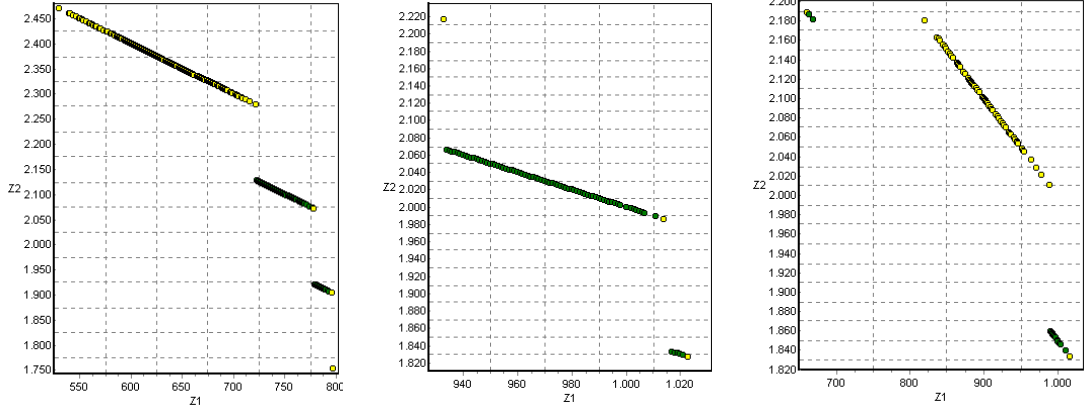


Figure 7: PF (Model 2) for three instances with $n = 30$

According to the assumption expressed in equation(8), the non-dominated solutions have the following properties:

3.1 Elementary properties

Proposition 5 Consider that x^+ is a solution of problem (9). If $|x^+| = n_1$ then $z_1(x^+) + z_2(x^+) = \alpha n_1$.

Proof. $z_1(x^+) + z_2(x^+) = \sum_{j=1}^n c_j^1 x_j^+ + \sum_{j=1}^n c_j^2 x_j^+ = \sum_{j=1}^n (c_j^1 + c_j^2) x_j^+ = \alpha \sum_{j=1}^n x_j^+ = \alpha n_1$. ■

The criteria space is composed of iso-item with the configuration of lines with slope -1, as illustrated in Figure 8.

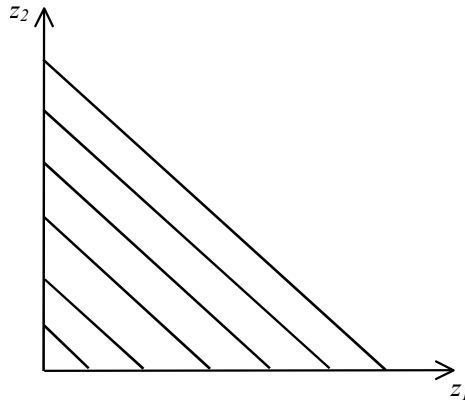


Figure 8: Structure of the criteria space

Proposition 6 If $z' = (z'_1, z'_2) \in ISO(n_i)$ and $z'' = (z''_1, z''_2) \in ISO(n_j)$, with $n_i \geq n_j \geq 0, n_i, n_j \in N$, then $z'_1 + z'_2 \geq z''_1 + z''_2$.

Proof. Once $z' \in ISO(n_i)$, $z'_1 + z'_2 = \alpha n_i$. Analogously, $z''_1 + z''_2 = \alpha n_j$. Since $n_i \geq n_j$, $\alpha n_i \geq \alpha n_j \Leftrightarrow z'_1 + z'_2 \geq z''_1 + z''_2$. ■

Proposition 7 Consider $z' = (z'_1, z'_2)$ and $z'' = (z''_1, z''_2)$. If $z', z'' \in ISO(n_j)$ then there is no dominance between z' and z'' .

Proof. Let us suppose by contradiction that z'' dominates z' . Then, $z''_1 \geq z'_1$ and $z''_2 \geq z'_2$, with $z'' \neq z'$. Consequently, $z''_1 + z''_2 > z'_1 + z'_2$. But, because z' and z'' belong to $ISO(n_j)$, it comes that $z'_1 + z'_2 = z''_1 + z''_2$. Thus, z'' cannot dominate z' . The proof that z' does not dominate z'' is similar. ■

Proposition 8 Consider $z' = (z'_1, z'_2)$ and $z'' = (z''_1, z''_2)$. If $z' \in ISO(n_i)$ and $z'' \in ISO(n_j)$ with $n_i > n_j$ then z'' does not dominate z' .

Proof. Let us suppose by contradiction that z'' dominates z' . Then, $z''_1 \geq z'_1$ and $z''_2 \geq z'_2$, with $z'' \neq z'$. So, $z''_1 + z''_2 > z'_1 + z'_2$. But, once $n_i > n_j$ it comes that $z'_1 + z'_2 > z''_1 + z''_2$. Consequently, z'' do not dominates z' . ■

$$\text{Let } n_p = \{ \max \sum_{j=1}^n x_j : \sum_{j=1}^n w_j x_j \leq W, x_j \in \{0, 1\}, j = 1, \dots, n \}.$$

Proposition 9 If x^+ is a feasible solution such that $z(x^+) \in ISO(n_p)$, for $n_1 < \dots < n_j < \dots < n_p$, then $z(x^+)$ is a non-dominated solution of problem (9).

Proof. As $n_p > n_j, j < p$ then from Proposition 7 all the solutions belonging to $ISO(n_p)$ are mutually non-dominated, and from Proposition 8, there is no solution on $ISO(n_j), j < p$ ($n_1 < \dots < n_j < \dots < n_p$) that dominates solutions on $ISO(n_p)$. Consequently, all the solutions on $ISO(n_p)$ are non-dominated solutions of problem (9). ■

The propositions above revealed a criteria space highly structured:

- solutions with the same cardinality are located in the same regions;
- solutions with the same cardinality do not dominate each other;
- solutions with lower cardinality cannot dominate solutions with higher cardinality;
- when moving from the upper right corner to the origin of the criteria space, the cardinality of the solutions decreases.

As a result of the propositions above, it is difficult for a solution with low cardinality to be an efficient one. Indeed, this difficulty increases as the cardinality decreases. Every solution belonging to a certain iso-item line excludes from further analysis an increasing region in iso-item with lower order. The knowledge of the maximum values on each criterion permits the definition of a lower bound on the cardinality of solutions. The proposition that follows enables the exclusion of a set of iso-item lines.

Proposition 10 If $x^{1*} = \arg \max \{c^1(x) : wx \leq W, x \in \{0, 1\}^n\}$ and $x^{2*} = \arg \max \{c^2(x) : wx \leq W, x \in \{0, 1\}^n\}$, then there is no efficient solution, x^+ , such that

$$|x^+| \leq \left\lfloor \frac{z_2(x^{1*}) + z_1(x^{2*})}{\alpha} \right\rfloor.$$

Proof. According to the definition of a non-dominated solution, the region where those solutions can be found is the rectangle defined by the points $z(x^{2*}); (z_1(x^{2*}), z_2(x^{1*})); z(x^{1*}); (z_1(x^{1*}), z_2(x^{2*}))$. Thus, all the iso-item such that $z_1(x^+) + z_2(x^+) \leq z_2(x^{1*}) + z_1(x^{2*})$ can be excluded.

As $z_1(x^+) + z_2(x^+) = \alpha |x^+|$, the later inequality is equivalent to $|x^+| \leq \frac{z_1(x^+) + z_2(x^+)}{\alpha}$ and so, due to the discrete nature of $|x^+|$, it comes that $|x^+| \leq \left\lfloor \frac{z_2(x^{1*}) + z_1(x^{2*})}{\alpha} \right\rfloor$. ■

Proposition 11 Let z^t and z^{t+1} be two consecutive feasible solutions in the criteria space, such that $z^t, z^{t+1} \in ISO(n_j)$ and $z_1^t > z_1^{t+1}$. If $z_1^t + z_2^{t+1} \geq \alpha n_{j-1}$ (with $n_{j-1} < n_j$) then there is no efficient solution, x^+ , such that $z_1(x^+) < z_1^{t+1}, z_2(x^+) < z_1^{t+1}$.

Proof. Suppose that x^+ is an efficient solution such that $z_1^t < z_1(x^+) < z_1^{t+1}, z_2^{t+1} < z_2(x^+) < z_2^t$ and $z(x^+) \in ISO(n_\tau), \tau < j$. Then, $z_1^t + z_2^{t+1} < \alpha n_\tau \leq \alpha n_{j-1}$, which contradicts the fact that $z_1^t + z_2^{t+1} \geq \alpha n_{j-1}$. ■

Corollary 1 If $\{z^{1t}, z^{2t}, \dots, z^{kt}\}$ is the set of non-dominated solutions available on $ISO(n_t)$ $t = p, p-1, \dots, q$ ordered according to increasing values of criterion z_1 , then, all the non-dominated solutions, (z_1^+, z_2^+) , belonging to $ISO(n_{q-1})$ observe

$$z_1^+ \in D^{n_{q-1}} \equiv \bigcup_{i=1}^{k-1} \left[z_1^{it}, \xi^{it} \right] \text{ if } z_1^{it} + z_2^{(i+1)t} < \alpha n_{q-1}, \text{ with } \xi^i = \alpha n_{q-1} - z_2^{(i+1)t}.$$

It is also interesting to note that,

Remark 4 All pairs of items are mutually non-dominated concerning the criteria coefficients, as the decrease on one criterion is deterministically accompanied by the increase in the other.

Despite Remark 4 there are items dominated by others depending on the value of the weights in the knapsack constraint. So, the following proposition, equivalent to Proposition 2, can be established.

Proposition 12 If x^+ is an efficient solution of (9) and if $c_j^1 > c_i^1$ and $w_j < w_i$, i.e., $\left((c_j^1, -w_j) \right)$ strictly dominates $(c_i^1, -w_i)$, $i, j \in \{1, \dots, n\}$ then $x_j^+ \geq x_i^+$.

Proof. Similar to the proof of Proposition 2. ■

3.2 Outline of some important aspects when computing the entire set of efficient solutions

Let us start by stating the following remarks.

Remark 5 For Model 2 there is no guarantee that all the extreme points of the $\text{Conv}(\{x \in \{0, 1\}^n : w^T x \leq W\})$ are integer points (see Figure 9 for an illustration).

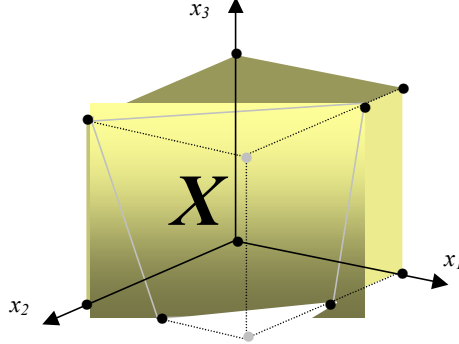


Figure 9: Non-integer extreme points in the decision space

Remark 6 The operation which requires the lowest number of changes in the value of the items, is not always associated with the determination of the nearest solution in the criteria space.

Nevertheless, it is interesting to note that despite that problem (9) could have non-supported solutions, the problem related with each iso-item line (obtained by adding $x^T x = \ell$, $\ell \in \{\text{min cardinality}, \dots, \text{max cardinality}\}$ to the set of constraints in (9)) has only supported ones. This feature makes easier the determination of the efficient solutions.

It can be seen that a point belonging to the set $\bar{X} \equiv \{x \in \{0, 1\}^n : |x| = \ell, wx \leq W\}$ is an extreme point of $\text{Conv}(\bar{X})$. Hence, when solving $\max \{z_1(x), z_2(x) : x \in \text{Conv}(\bar{X})\}$ the integer solutions are also obtained. Representing by $\Theta^{(2,\ell)}$ the set of efficient solutions of the later problem we have the following result, consequence of the linear multiple criteria programming:

Remark 7 The set $\Theta^{(2,\ell)}$, is connected.

Consequently the set \bar{X} belongs to a set which is connected. However, this aspect does not mean that $\bigcup_{\ell} \Theta^{(2,\ell)}$ is itself connected.

A possible procedure for solving (9) would be considering each cardinality separately and, for each of them, computing the set of feasible solutions. Taking into account the propositions and remarks above, starting with the *maxcardinality* is the best strategy, since the *mincardinality* can be updated (increased) as new solutions are found. A filtering process assures that only efficient solutions are considered. Proposition 7 assures that the filtering process is only necessary with solutions with different cardinalities.

The problem to be solved can thus be stated as the determination of all the optimal solutions of the following binary model:

$$\begin{aligned}
& \max \sum_{j=1}^n x_j \\
& s.t. : \\
& \sum_{j=1}^n w_j x_j \leq W \\
& \sum_{j=1}^n x_j \leq \ell \\
& x_j \in \{0, 1\}, j = 1, \dots, n
\end{aligned} \tag{10}$$

with $\ell \in \{\min \textit{cardinality}, \dots, \max \textit{cardinality}\}$.

This problem is a special case of the single criteria $\{0,1\}$ -knapsack problem with cardinality constraints, which is considered by Pisinger (1998) and Caprara *et al.* (2000). In these papers, the problem is solved through dynamic programming where the recursive equations are proposed.

Using the terminology of dynamic programming we can define for problem (10) a state $\widehat{h}(j, b, \eta)$ and the recursive equation in the following way:

- State equation

$$\widehat{h}(j, b, \eta) = \left\{ x \in \{0, 1\}^n : \sum_{t=1}^j w_t x_t = b, \sum_{t=1}^j x_t = \eta, x_{j+1} = \dots = x_n = 0, j \geq \eta \right\} \tag{11}$$

with $j = 1, \dots, n; \eta = 1, \dots, k; b = 1, \dots, W$.

- Recursive equation

It is considered the minimum used capacity for a given pair (j, η) to define the recursive equation:

$$h(j, \eta) = \min \left\{ \sum_{t=1}^j w_t x_t : \sum_{t=1}^j x_t = \eta \right\} \tag{12}$$

Thus,

$$h(j, \eta) = \min \{h(j-1, \eta), h(j-1, \eta-1) + w_j\} \tag{13}$$

All the optimal solutions of problem (10) are associated with the state $\widehat{h}(n, b, k)$ such that $h(n, k) \leq W$.

The constraint $\sum_{j=1}^n x_j \leq \ell$ can be used to greatly simplify the network used to represent the dynamic programming process: a node can be linked to the terminal node as long as 1) there is ℓ items included; 2) it is impossible to achieve a solution with ℓ items (in this case the corresponding path can be discarded).

4 Model 3: Constant sum and equal weighted items

This model results from the combination of Models 1 and 2. Thus, it is assumed that,

$$\begin{aligned} c_j^1 + c_j^2 &= \alpha, j = 1, \dots, n \\ w_j &= k, j = 1, \dots, n \end{aligned} \quad (14)$$

Replacing constraints (14) in problem (1) gives:

$$\begin{aligned} \max z_1(x_1, \dots, x_j, \dots, x_n) &= \sum_{j=1}^n c_j^1 x_j \\ \max z_2(x_1, \dots, x_j, \dots, x_n) &= \sum_{j=1}^n (\alpha - c_j^1) x_j \\ \text{s.t. :} \\ \sum_{j=1}^n x_j &\leq \left\lfloor \frac{W}{k} \right\rfloor = \ell \\ x_j &\in \{0, 1\}, j = 1, \dots, n \end{aligned} \quad (15)$$

Figure 10, presents the PF of problem (15). It is a single linear iso-item line with slope -1.

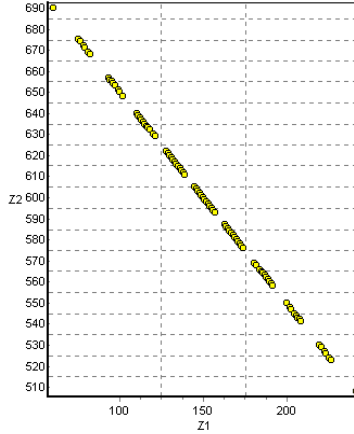


Figure 10: PF (Model 3) of an instance with $n = 10$ and $\ell = 5$

Constraints $c_j^1 + c_j^2 = \alpha, j = 1, \dots, n$ give rise to a criteria space composed of linear iso-item with slope -1 (as in Model 2), and constraints $w_j = k, j = 1, \dots, n$ convert the capacity constraint into a cardinality one (as in Model 1).

4.1 Elementary properties

Combining these two features, it comes that all the feasible solutions have the same cardinality and simultaneously are efficient ones, as stated in the proposition below.

Proposition 13 All the feasible solutions, x^+ , such that $|x^+| = \ell$, with $\ell = \left\lfloor \frac{W}{k} \right\rfloor$, are efficient solutions of problem (15).

Proof. Since every solution with cardinality lower than ℓ cannot be efficient, it is only needed to prove that there is no dominance between solutions with cardinality ℓ . Since this result comes from Proposition 7, it is proved that all the feasible solutions with cardinality ℓ are efficient. ■

Considering the structure of the criteria coefficients and the fact that all the weights in the knapsack constraint are equal, we can state the following remark:

Remark 8 All the pairs of items of problem (15) are mutually non-dominated. The dominance relation has only one layer without any edges between nodes.

For problem (15) and according to Proposition 7 it is then possible to establish the total number of efficient solutions, hence it is equal to the number of possible combinations that are possible to be generated with ℓ 1's in n positions. This number is thus ${}^n C_\ell = \frac{n!}{(n-\ell)!\ell!}$.

For instance, if a problem has only 20 items and $\ell = 10$, then there are 184.756 efficient solutions, which is, indeed, a very high number when compared with the average number of solutions of randomly generated instances without constraints (14).

The problem that has a maximum number of efficient solutions is obtained when $\ell = \left\lfloor \frac{n-1}{2} \right\rfloor$ or $\ell = \left\lceil \frac{n-1}{2} \right\rceil$, *i.e.*, with approximately 50% of the items.

4.2 Outline of some important aspects when computing the entire set of efficient solutions

The computation of the entire set of efficient solutions consists of determining the entire set of feasible solutions, *i.e.*, solutions with cardinality ℓ . Once all the feasible solutions are efficient, the key point in solving problem (15) is the development of a transition rule between solutions, particularly, a rule which starts with a solution that maximizes one of the two criteria and achieves a solution that maximizes the other one. Due to the deterministic relation between criteria, this issue can be stated as the problem of ranking all the feasible values of criterion z_1 or z_2 , which are associated with $|x| = \ell$.

An important starting point is to investigate whether changing the value of the smallest number of variables leads to the nearest solution in the criteria space. Unfortunately, the result is negative.

Remark 9 Even in this highly structured model (problem (15)) there is no direct correspondence between Euclidean distances in the criteria and decision spaces, as it can be seen in the following instance:

$$\begin{aligned} \text{Max } z_1(x) &= x_1 + 16x_2 + 24x_3 + 32x_4 + 6x_5 \\ \text{Max } z_2(x) &= 149x_1 + 134x_2 + 126x_3 + 118x_4 + 144x_5 \\ \text{s.t.:} \end{aligned}$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 3$$

$$x_j \in \{0, 1\}, j = 1, \dots, 5.$$

The instance has 10 efficient solutions. In Figure 11 each node i ($i=1, \dots, 10$) corresponds to the efficient solution i , where the solutions are ordered in the criteria space. If two nodes are linked then it means that in the decision space it is only necessary to change the value of two variables to move from one solution to the other. It can be observed that there are solutions which are adjacent in the criteria space but in the decision space the distance between them is not the smallest.

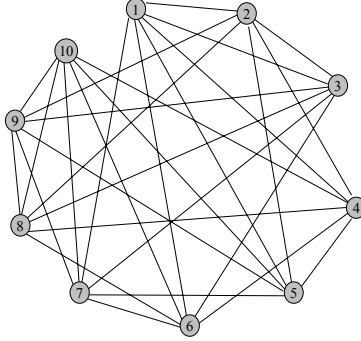


Figure 11: Set of non-dominated solutions

The remark above put aside an organized descendent procedure for obtaining the efficient solutions.

Again, the ranking procedure by Murty (1983) can be used as an alternative due to the following remarks:

Remark 10 All the extreme points of the feasible region $\text{Conv}(\overline{X})$ are integer points (as in Model 1), thus the number of extreme points of $\text{Conv}(\overline{X})$ is ${}^n C_\ell$, and each extreme point is connected with $\ell \times (n - \ell)$ extreme points. This is the number of possible directions of leaving an extreme solution and achieving another one.

Let us represent by $\Theta^{(3)}$ the set of efficient solutions of problem (15). As all the efficient solutions of the problem are supported ones, then it comes from the multiple criteria linear programming the following result:

Remark 11 The set $\Theta^{(3)}$ is connected.

Moving between adjacent efficient solutions can be achieved by using the simplex method with a single criterion (z_1 or z_2) and bounded variables. Like in Section 2.2 the simplex tableau has only one basic variable, and the line of reduced prices is just the difference between the coefficient of variables in the objective function and the value of basic variable. Suppose that criterion z_2 was selected. Thus, from a given solution, the value of the objective function does not decrease when a non-basic variable, x_j , enters the basis, when $c_j^2 - c_f^2 \geq 0$ and $x_j = 0$, or when $c_j^2 - c_f^2 \leq 0$ and $x_j = 1$. In the first case it is only obtained a different basis for the same extreme point, while in the second case, and if $c_j^2 - c_f^2 < 0$, a new adjacent point is obtained.

According to the ranking procedure by Murty (1983), from a given point, all the adjacent extreme points, that do not decrease the value of the objective function, have to be determined. The procedure ends when it is not possible to do this from any available point.

5 Extensions to general multiple criteria $\{0,1\}$ -problems

In the general bi-criteria and multiple criteria problems there are no restrictive conditions on the coefficients in the model. Despite that, all these problems can be converted into other problems where conditions of Model 2 hold. This can be done by using a very interesting transformation, that is, the incorporation of an additional "fictitious" criterion.

To show this, let us consider the bi-criteria case. If $c_j^1 + c_j^2 = \alpha, j = 1, \dots, n$ does not hold (assuming $c_j^1, c_j^2 > 0$) that, then if we make $\alpha = \max_{j=1, \dots, n} \{c_j^1 + c_j^2\}$ a third criterion can be defined as follows:

$$c_j^3 = \alpha - (c_j^1 + c_j^2), j = 1, \dots, n \quad (16)$$

With the "fictitious" criterion the immediate result is that $c_j^1 + c_j^2 + c_j^3 = \alpha, j = 1, \dots, n$. Consequently, all the propositions derived in Section 3 can be extended to the case where there are 3 criteria.

Feasible solutions of the three-criteria problems are located in planes, as illustrated in Figure 12. In this figure the $ISO(\delta), \delta \in \{n_q, n_{q-1}, n_{q-2}\}$, with $n_q > n_{q-1} > n_{q-2}$ are planes.

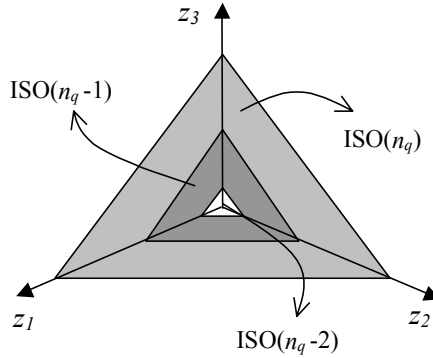


Figure 12: Iso-item planes

The transition between two solutions in the same iso-plane is also made by changing a balanced number of variables.

Since Model 2 is independent from the structure of the feasible set, all the derived results can be applied to the general $\{0,1\}$ -multiple criteria problems.

6 Conclusions

In this paper, three particular models of the bi-criteria $\{0,1\}$ -knapsack were built. Properties about the configuration and the structure of the PF were devised. The three models are related to each other by the important concept of cardinality of the set of efficient solutions. In the first model, all the efficient solutions have the same cardinality and the configuration of the PF is a "smooth" concave function. In the second model, the cardinality of efficient solutions brings about a PF composed by lines of slope -1. In the third model, the PF is a single line of slope -1: all the efficient solutions have the same cardinality. With respect to this model, the connectedness of the efficient solutions set was observed.

Despite the restrictive formulations, in the previous section it was shown that general bi-criteria (and also multiple criteria) $\{0,1\}$ -problems can be transformed in such a way that conditions of Model 2 hold.

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