Instituto de Engenharia de Sistemas e Computadores de Coimbra
Institute of Systems Engineering and Computers
INESC - Coimbra

Maria do Céu Marques and Joana Matos Dias

Simple dynamic location problem with uncertainty: a primal-dual heuristic approach

No. 14 2011

ISSN: 1645 – 2631

Instituto de Engenharia de Sistemas e Computadores de Coimbra
INESC - Coimbra
Rua Antero de Quental, 199; 3000 – 033 Coimbra; Portugal
www.inescc.pt

Simple dynamic location problem with uncertainty: a primal-dual heuristic approach

Maria do Céu Marques* and Joana Matos Dias†

Abstract

In this work the simple dynamic facility location problem is extended to uncertain realizations of the potential locations for facilities and the existence of customers as well as fixed and variable costs. With limited knowledge about the future, a finite and discrete set of scenarios is considered. The decisions to be made are where and when to locate the facilities, and how to assign the existing customers over the whole planning horizon and under each scenario, in order to minimize the expected total costs. The location decisions have to take into account all possible scenarios, but cannot be changed according to each scenario in particular, which make the problem non–separable (and more difficult). A mixed linear programming formulation is shown, and a primal-dual heuristic is developed.

keywords: Dynamic Location Problems, Primal-Dual Heuristic, Uncertainty, Scenarios

1 Introduction

Location problems, most of the times, are strategic decisions that are costly to revert and that have consequences in the medium and long term. So, when making a location decision, the Decision Maker should consider not only the present situation, but also the future. According to Erlenkotter (1981), two main characteristics force the consideration of a dynamic location problem: the assignment costs change significantly during the planning horizon, and there must be significant costs for relocating facilities. If the first characteristic is absent, then the problem can be formulated as the uncapacitated facility location problem (UFLP); if the second characteristic is absent, then a set of disconnected UFLP can be considered (one for each period of the planning horizon). It is not possible, however, to think beyond the present and to consider a planning horizon without realizing that decisions are to be taken in an uncertain environment.

In this research report we consider a dynamic location problem where uncertainty is explicitly incorporated, represented by a finite and discrete set of future scenarios. Fixed and assignment costs are scenario dependent, as well as the set of customers and the set of potential locations for facilities. We formulate our problem as an integer linear program, that contains the UFLP and the dynamic uncapacitated facility location (DUFLP) as particular problems. As well known, the UFLP and its generalizations are NP-hard (Cornuejols et al., 1990). We propose a new primal–dual heuristic directly inspired on the approaches developed by Erlenkotter (1978) and Van Roy and Erlenkotter (1982), designed for the static and dynamic versions of the UFLP, respectively.

Several authors have considered location problems with uncertainty. Louveaux and Peeters (1992) consider a stochastic UFLP in which demands, variable production and transportation costs, and selling prices (incorporated in the model) can be random. The stochastic problem is formulated

*Instituto Superior de Engenharia de Coimbra, Instituto Politécnico de Coimbra, Rua Pedro Nunes, 3030-199 Coimbra, Portugal, *cmarques@isec.pt*.

†Faculdade de Economia and INESC-Coimbra, Universidade de Coimbra, Av. Dias da Silva, 165, 3004-512 Coimbra, Portugal, *joana@fe.uc.pt*.

as a two–stage stochastic program with recourse, where the first–stage decisions are the location and size of the facilities to be established, and the second–stage decisions are the allocation of the available production to the most profitable demands. As opposed to the deterministic case, the choice of both the demands to be served and the size of the facilities to be established also becomes part of the decision process. The authors also propose a heuristic based in Erlenkotter (1978). Laporte et al. (1994) consider a stochastic Capacitated Facility Location Problem. The problem consists of optimally determining the location and size of facilities given that future customer demand is uncertain. The problem can also be viewed as a two–stage stochastic integer program. Serra and Marianov (1998) consider a p-median based model in which travel times between nodes and/or demand at nodes are uncertain. In this paper uncertainty is treated by the scenario approach. Two p-median formulations are presented, the minmax and the regret approaches. The authors propose a heuristic method for both formulations, and a real application to the location of fire stations in Barcelona is presented. More recently, Ravi and Sinha (2004) also propose a two-stage stochastic version of the UFLP and an approximation algorithm to solve it. Here, demand and fixed costs are both random, and facilities may be opened in either the first or second stage. Lin (2009) proposes a stochastic version of the single–source capacitated facility location problem in which the demand is uncertain. The objective function is to minimize the total system costs including fixed facility costs and costs of servicing each demand point by its assigned facility. Simultaneously, recognizing that facilities should provide an adequate level of service, the model also incorporates facility service level requirements, being the probability that each open facility can cope with the stochastic demand assigned. The set of demand quantities are assumed to be independent random variables. In a dynamic framework, we refer, for instance, to Jornsten and Bjorndal (1994), Current et al. (1997) and Romauch and Hartl (2005). Jornsten and Bjorndal (1994) also consider the DUFLP under uncertainty, where the fixed and variable costs are described via a set of scenarios. To solve the dynamic and stochastic program, the authors use the scenario and policy aggregation described by Rockafellar and Wets (1991). Current et al. (1997) address dynamic location problems in which the total number of facilities to be sited over the planning horizon is uncertain. Two decision criteria are considered: the minimization of the maximum regret and the minimization of expected opportunity loss. Under the decision criteria, each problem locates an initial number of facilities when the total number is unknown. Romauch and Hartl (2005) consider a dynamic facility location problem with uncertain demand, described by scenarios. The problem seeks the optimal decisions for production, inventory and transportation, to serve the costumers during a fixed number of periods. It is assumed that the production sites have limited storage capacities. The model is first solved by dynamic programming and then a heuristic is proposed, the Sample Average Approximation Method (SSA) adapted to the multi-period case. For other references and extensive reviews on location problems under uncertainty we refer to (Louveaux, 1993; Snyder, 2006).

In the following section the notation used in this research report is introduced and our problem is described. In section 3 the corresponding dual problem is presented and the heuristic is described. Section 4 presents two illustrative examples and finally section 5 includes conclusions and future work directions.

2 Notation and Description of the Problem

Consider a planning horizon represented by a discrete set of time periods $\mathcal{T} = \{1, ..., t, ..., T\}$. The *future* will be one of a finite set of possibilities, represented by a discrete set of *scenarios* $\mathcal{S} = \{1, ..., s, ..., S\}$, where each scenario characterizes the value of all uncertain elements. Suppose that each $s \in \mathcal{S}$ will occur with probability p^s such that $\sum_{s\in\mathcal{S}} p^s = 1$.

Let the set of potential facility sites be denoted by $J = \{1, ..., j, ..., M\}$ and the set of possible costumers locations (or demand points) by $I = \{1, ..., i, ..., N\}$. In reality, these sets include all the potential facility locations and all the potential customers for all possible scenarios, despite the fact that for each scenario in particular possibly only a subset of potential locations and a subset of customers is considered. The reason for this is that we consider uncertainty associated not only with the fixed and variable costs, but also associated with the existence of customers and the future existence of potential locations. Only at the first time period are these sets equal for all scenarios (the present situation).

Let us define δ_{it}^s as equal to 1 if customer i exists during period t for scenario s, and 0 otherwise. Then we have to guarantee that all customers such that $\delta_{it}^s = 1$ are assigned to an open facility, for all $(t, s) \in \mathcal{T} \times \mathcal{S}$. In terms of costs, the model considers not only fixed costs (opening and operating), but also variable costs associated with the assignment of customers to the facilities. For $(j, t, s) \in J \times \mathcal{T} \times \mathcal{S}$, let f_{jt}^s be the fixed cost of establishing (opening) facility j at the beginning of period t plus the operating and subsequent costs in period t, under scenario s; for $(i, j, t, s) \in I \times J \times \mathcal{T} \times \mathcal{S}$, c_{ijt}^s represents the assignment cost of customer i to facility j in period t and under scenario s. If it is not possible to open facility j at the beginning of time period t, under scenario s, then the corresponding fixed cost will be considered equal to $+\infty$. We assume that once a facility is opened, it stays open until the end of the planning horizon.

The decisions to be made are where and when to locate new facilities, and how to assign the existing customers over the whole planning horizon and under each scenario. Thus, we define the following binary decision variables for the problem: x_{jt} equals 1 if facility j is opened at the beginning of period t, and 0 otherwise; y_{ijt}^s equals 1 if customer i is assigned to facility j in period t and under scenario s, and 0 otherwise. As a matter of fact, assignment decisions are considered to be taken a period at a time, so they can be changed according to the scenario that came true. Location decisions are hard to revert, so we have to live with the decision taken whatever the scenario that came to occur. Our aim is to make the best location decisions, considering the uncertainty associated with the future. Several different objective functions could be considered, but for now we consider the minimization of expected total costs (fixed and assignment costs).

We can formulate the problem as follows:

(P)

$$\min \sum_{t \in \mathcal{T}} \sum_{j \in J} \sum_{s \in \mathcal{S}} p^s f_{jt}^s x_{jt} + \sum_{s \in \mathcal{S}} \sum_{t \in \mathcal{T}} \sum_{i \in I} \sum_{j \in J} p^s c_{ijt}^s y_{ijt}^s \tag{2.1}$$

subject to

$$\sum_{j \in J} y_{ijt}^s = \delta_{it}^s, \ \forall\ i \in I, t \in \mathcal{T}, s \in \mathcal{S} \tag{2.2}$$

$$\sum_{\tau=1}^{t} x_{j\tau} - y_{ijt}^s \geq 0, \ \forall\ i \in I, j \in J, t \in \mathcal{T}, s \in \mathcal{S} \tag{2.3}$$

$$\sum_{t \in \mathcal{T}} (-x_{jt}) \geq -1, \ \forall\ j \in J \tag{2.4}$$

$$x_{jt} \in \{0, 1\}, \ \forall\ j \in J, t \in \mathcal{T} \tag{2.5}$$

$$y_{ijt}^s \in \{0, 1\}, \ \forall i \in I, j \in J, t \in \mathcal{T}, s \in \mathcal{S} \tag{2.6}$$

The objective function (2.1) minimizes the expected total costs (fixed plus variable costs). Constraints (2.2) require that, under each scenario and in every time period, an existing customer be assigned to exactly one facility. Constraints (2.3) impose that an existing customer can only be assigned to open facilities. A customer can be assigned to different facilities at different time periods and different scenarios. Constraints (2.4) ensure that each facility is opened at most once during the time horizon (located at the same site in all scenarios). Finally, (2.5)–(2.6) restrict the decision variables to be binary.

Formulation **(P)** contains the UFLP ($|\mathcal{T}| = |\mathcal{S}| = 1$) and the DUFLP ($|\mathcal{T}| > 1, |\mathcal{S}| = 1$) as particular problems, and has $|J|\,|\mathcal{T}|+|J|\,|I|\,|\mathcal{T}|\,|\mathcal{S}|$ binary variables and $|I|\,|\mathcal{T}|\,|\mathcal{S}|+|J|\,|I|\,|\mathcal{T}|\,|\mathcal{S}|+|J|$ restrictions (not counting the zero-one constraints). Even for moderate dimensions of these sets, **(P)** becomes a quite large integer linear program.

3 Heuristic Approach

We propose a primal-dual heuristic based on the approaches developed by Erlenkotter (1978) and Van Roy and Erlenkotter (1982) for the static and dynamic versions of the UFLP, respectively. The main idea of the approach is to obtain good solutions from the corresponding dual problem, more precisely from the so-called condensed dual problem. The various procedures are designed to reduce the duality gap between dual and primal function values. The dual ascent procedure constructs a dual solution and an associated set of candidate facility locations. The primal procedure yields a corresponding candidate primal solution. If the dual and primal solutions satisfy all complementary slackness conditions, then the solutions are optimal. If not, the heuristic continues with the adjustment procedures in order to improve these solutions.

In order to describe the heuristic, we begin by formulating the dual problem, the condensed dual problem and the complementary slackness conditions between the dual problem and (P).

3.1 Dual Problem and Complementary Slackness Conditions

Consider the primal **(P)**. Defining in (2.1) $\mathcal{C}^s_{ijt} = p^s c^s_{ijt}$ and $\mathcal{F}^s_{jt} = p^s f^s_{jt}$, and considering dual variables v^s_{it}, w^s_{ijt} and u_j associated with the restrictions (2.2), (2.3) and (2.4), respectively, the dual of **(P)** is given by:

(D)

$$\max \sum_{i\in I}\sum_{t\in\mathcal{T}}\sum_{s\in\mathcal{S}} \delta^s_{it}\, v^s_{it} - \sum_{j\in J} u_j \tag{3.7}$$

subject to

$$v^s_{it} - w^s_{ijt} \leq \mathcal{C}^s_{ijt},\ \forall\ i\in I, j\in J, t\in\mathcal{T}, s\in\mathcal{S} \tag{3.8}$$

$$\sum_{i\in I}\sum_{s\in\mathcal{S}}\sum_{\tau=t}^{T} w^s_{ij\tau} - u_j \leq \sum_{s\in\mathcal{S}} \mathcal{F}^s_{jt},\ \forall\ j\in J, t\in\mathcal{T} \tag{3.9}$$

$$w^s_{ijt} \geq 0,\ \forall\ i\in I, j\in J, t\in\mathcal{T}, s\in\mathcal{S} \tag{3.10}$$

$$u_j \geq 0,\ \forall j\in J \tag{3.11}$$

We may set

$$w^s_{ijt} = \max\{0, v^s_{it} - \mathcal{C}^s_{ijt}\},\ \forall\ i,j,t,s, \tag{3.12}$$

to obtain the condensed dual problem:

(D)

$$\max \sum_{i\in I}\sum_{t\in\mathcal{T}}\sum_{s\in\mathcal{S}} \delta^s_{it}\, v^s_{it} - \sum_{j\in J} u_j \tag{3.13}$$

subject to

$$\sum_{i\in I}\sum_{s\in\mathcal{S}}\sum_{\tau=t}^{T} \max\{0, v^s_{i\tau} - \mathcal{C}^s_{ij\tau}\} - u_j \le \sum_{s\in\mathcal{S}} \mathcal{F}^s_{jt},\ \forall\ j,t \tag{3.14}$$

$$u_j \ge 0,\ \forall j \tag{3.15}$$

The corresponding slack variables π_{jt} for constraints (3.14) are given by:

$$\pi_{jt} = \sum_{s\in\mathcal{S}} \mathcal{F}^s_{jt} - \sum_{i\in I}\sum_{s\in\mathcal{S}}\sum_{\tau=t}^{T} \max\{0, v^s_{i\tau} - \mathcal{C}^s_{ij\tau}\} + u_j,\ \forall\ j,t. \tag{3.16}$$

Then, the complementary slackness conditions are:

$$\pi_{jt}\, x_{jt} = 0, \quad \forall j,t \tag{3.17}$$

$$v^s_{it}\left(\sum_j y^s_{ijt} - \delta^s_{it}\right) = 0, \quad \forall\ i,t,s \tag{3.18}$$

$$w^s_{ijt}\left(\sum_{\tau=1}^{t} x_{j\tau} - y^s_{ijt}\right) = 0, \quad \forall\ i,j,t,s \tag{3.19}$$

$$u_j\left(1 - \sum_t x_{jt}\right) = 0, \quad \forall\ j \tag{3.20}$$

$$y^s_{ijt}\ \left(v^s_{it} - \mathcal{C}^s_{ijt} - w^s_{ijt}\right) = 0, \quad \forall\ i,j,t,s \tag{3.21}$$

3.2 Primal-Dual Heuristic

For ease in the exposition, let us reindex, for each scenario s, $\mathcal{C}^s_{ijt}$ for each (i,t) in nondecreasing order as $\mathcal{C}^{s(k)}_{it}$, for $k = 1, 2, ..., k^s_{it}$, where k^s_{it} denotes the number of facility-to-customer links for (i,t) under scenario s. Thus, $\mathcal{C}^{s(1)}_{it} = \min_{j\in J}\{\mathcal{C}^s_{ijt}\}$. For convenience, we also include $\mathcal{C}^{s(k^s_{it}+1)}_{it} = +\infty, \forall\ (i,t,s)$.

Let I^+ be the set of *pseudo customers* (i,t,s) corresponding to the dual variables v^s_{it} that the procedure will try to increase. Initially, I^+ will be equal to all possible combinations $(i,t,s) \in I\times\mathcal{T}\times\mathcal{S}$, except those such that $\delta^s_{it} = 0$. Later, I^+ will be set within the respective procedures. We note that a customer without demand does not contribute to the improvement of the dual objective function value and does not also contribute to any violation of the complementary slackness conditions. Thus, these customers are excluded from the ascent procedures.

The steps of the heuristic are as follows:

1. Set $v^s_{it} = \mathcal{C}^{s(1)}_{it}$, $\forall\ (i,t,s)$, and $u_j = 0, \forall\ j$.
   Set $I^+ = \{(i,t,s) \in I\times\mathcal{T}\times\mathcal{S} : \delta^s_{it} = 1\}$.

2. Execute the dual ascent procedure.
3. Execute the primal procedure. If an optimal solution is found, then stop.
4. Execute the primal–dual adjustment procedure as the dual objective function value increases. If an optimal solution is found, then stop.
5. Execute the dual adjustment procedure for u_j. If the dual solution is changed, then return to step 3.

3.2.1 Dual Ascent Procedure

This procedure will iteratively increase the values of variables v_{it}^s, decreasing the slacks' values accordingly.

In what follows, $(i,t,s)_q$, with $q \leq |I \times \mathcal{T} \times \mathcal{S}|$, represents a given, but arbitrary, sequence of pseudo customers.

1. Consider any dual feasible solution $\{v_{it}^s\}$ such that $v_{it}^s \geq \mathcal{C}_{it}^{s(1)}, \forall\ (i,t,s)$, and $\pi_{jt} \geq 0, \forall\ (j,t)$. For each (i,t,s) define $k(i,t,s) = \min\{k : v_{it}^s \leq \mathcal{C}_{it}^{s(k)}\}$. If $v_{it}^s = \mathcal{C}_{it}^{s(k(i,t,s))}$, then $k(i,t,s) \leftarrow k(i,t,s)+1$.
2. $(i,t,s) \leftarrow (i,t,s)_1$ and $q \leftarrow 1$; $r=0$.
3. If $(i,t,s) \notin I^+ \vee \delta_{it}^s = 0$, then go to step 7.
4. Set $\Delta_{it}^s = \min\limits_j\{\pi_{j\tau} : v_{it}^s - \mathcal{C}_{ijt}^s \geq 0, \tau \leq t\}$.
5. If $\Delta_{it}^s > \mathcal{C}_{it}^{s(k(i,t,s))} - v_{it}^s$, then $\Delta_{it}^s = \mathcal{C}_{it}^{s(k(i,t,s))} - v_{it}^s$; $r=1$; $k(i,t,s) \leftarrow k(i,t,s)+1$.
6. For all $j \in J$ with $v_{it}^s - \mathcal{C}_{ijt}^s \geq 0$, set $\pi_{j\tau} = \pi_{j\tau} - \Delta_{it}^s, \tau \leq t$; set $v_{it}^s = v_{it}^s + \Delta_{it}^s$.
7. If $q < |I^+|$, then $q \leftarrow q+1$, $(i,t,s) \leftarrow (i,t,s)_q$, and return to step 3.
8. If $r=1$, then return to step 2, otherwise stop.

3.2.2 Primal Procedure

From the dual ascent procedure results the dual feasible solution $\{v_{it}^{s+}\}$ with an objective function value v_D^+, and associated slacks $\{\pi_{jt}^+\}$. A corresponding primal feasible solution, $\{x_{jt}^+\}$ and $\{y_{ijt}^{s+}\}$, can be constructed, with an objective function value v_P^+.

In order to describe the primal procedure, let us first define the following sets:
$J^* = \{(j,t) \in J \times \mathcal{T} : \pi_{jt}^+ = 0\}$
$J_t^* = \{j \in J : (j,\tau) \in J^*, \tau \leq t\}, \forall t \in \mathcal{T}$
$J_t^+ = \{j \in J : \text{facility } j \text{ is open at time } t\}, \forall t \in \mathcal{T}$
Define $t_1(j) = \min\{\gamma : j \in J_\gamma^+\}$ and $t_2(j) = \max\{\gamma \leq t_1(j) : (j,\gamma) \in J^*\}$. Then,
$J^+ = \big\{(j, t_2(j)) \in J \times \mathcal{T} : j \in J_\tau^+ \text{ for some } \tau\big\}$.

The set J^* corresponds to all (j,t) such that j can be opened at the beginning of period t without violating (3.17); set J_t^* corresponds to all j that can be opened up to t; set J_t^+ corresponds to all j that are actually open during t; set $J^+ \subseteq J^*$ corresponds to all j that open at the beginning of t, i.e., J^+ dictates what facilities are actually opened and when (location decisions).

The facilities that are considered first are the ones that at a given time t should be assigned to a given customer (i,s), according to conditions (3.19), called *essential* facilities. Other facilities are only opened if strictly necessary. If a facility j needs to be open at some time period(s) and the first time period when it needs to be open is t, then it will be opened at the beginning of time period

$t_2(j)$, defined as being the time period closest and less than or equal to t such that the corresponding slack is equal to zero. It should be noted that, as we are dealing with an uncapacitated location problem, there will always be an admissible solution that can be built in this way: we can be sure that there exists at least one facility j such that π_{j1} is equal to zero (at least one facility can be opened at the beginning of the first time period). If this was not true, then it would still be possible to improve the dual solution by increasing at least one v^s_{i1} dual variable.

The steps of the primal procedure are as follows:

1. Set $J^+ = J^+_t = \emptyset$, $\forall t$. Build J^* and J^*_t, $\forall t$.
2. For each $t \in \mathcal{T}$, if $j \in J^*_t$ such that $\exists (i,s) : v^{s+}_{it} \geq \mathcal{C}^s_{ijt}$ and $v^{s+}_{it} < \mathcal{C}^s_{ij't}, \forall j' \in J^*_t \backslash \{j\}$, then $J^+_\tau = J^+_\tau \cup \{j\}$, $\forall \tau \geq t$.
3. For each (i,t,s), if $\nexists j \in J^+_t$ with $v^{s+}_{it} \geq \mathcal{C}^s_{ijt}$, then $J^+_\tau = J^+_\tau \cup \Big\{ j \in J^*_t : \mathcal{C}^s_{ijt} = \min\{\mathcal{C}^s_{ij't} : v^s_{it} \geq \mathcal{C}^s_{ij't}\} \Big\}$, $\forall \tau \geq t$.
4. Build J^+.
5. For each $j \in J$, if $u_j > 0$ and $\nexists t : (j,t) \in J^+$, then
$$J^+ = J^+ \cup \left\{ (j,t) \in J^* : \sum_{s \in \mathcal{S}} \mathcal{F}^s_{jt} = \min_{(j,\tau) \in J^*} \left\{ \sum_{s \in \mathcal{S}} \mathcal{F}^s_{j\tau} \right\} \right\}.$$
6. Update J^+_t, $\forall t$. Assign each (i,t,s) to facility $j \in J^+_t$ with lowest $\mathcal{C}^s_{ijt}$.

Step 5 tries to minimize the violation of conditions (3.20). If there is a facility j such that $u_j > 0$ and $\sum_t x_{jt} \neq 1$, then j has to be opened in some time period. We chose to open the facility at the beginning of time period t such that the total weighted fixed open cost is minimized, while respecting conditions (3.17).

3.2.3 Primal–Dual Adjustment Procedure

The primal–dual adjustment procedure will try to enforce the conditions (3.19) that can still be violated.

Consider the additional sets and definitions:
$J^{s*}_{it} = \{j : \exists \tau \leq t \mid (j,\tau) \in J^* \text{ and } v^s_{it} \geq \mathcal{C}^s_{ijt}\}$, $\forall (i,t,s)$
$J^{s+}_{it} = \{j : \exists \tau \leq t \mid (j,\tau) \in J^+ \text{ and } v^s_{it} > \mathcal{C}^s_{ijt}\}$, $\forall (i,t,s)$
$I^+_{jt} = \{(i,\tau,s) : J^{s*}_{i\tau} = \{j\} \text{ for } \tau \geq t\}$, $\forall (j,t)$

A best source and a second-best source for (i,t,s) in J^+_t are denoted by $j(i,t,s)$ and $j'(i,t,s)$, respectively: $\mathcal{C}^s_{ij(i,t,s)t} = \min_{j \in J^+_t} \{\mathcal{C}^s_{ijt}\}, \forall (i,t,s)$;
$\mathcal{C}^s_{ij'(i,t,s)t} = \min_{j \in J^+_t, j \neq j(i,t,s)} \{\mathcal{C}^s_{ijt}\}, \forall (i,t,s)$ for $|J^{s+}_{it}| > 1$
$\mathcal{C}^{s-}_{it} = \max_j \{\mathcal{C}^s_{ijt} : v^s_{it} > \mathcal{C}^s_{ijt}\}$

If $|J^{s+}_{it}| > 1$, for some (i,t,s), then a complementary slackness condition (3.19) is violated. In such case, the decrease of the variable v^s_{it} causes the increase of at least two slacks $\pi_{j\tau}$, associated with distinct facilities. Set I^+_{jt} corresponds to all variables $v^s_{i\tau}$ whose value can be increased with the increase of slacks $\pi_{j\tau}, \tau \leq t$.

1. $(i,t,s) \leftarrow (i,t,s)_1, q \leftarrow 1$; set $v_D = v^+_D$ and $v_P = v^+_P$; set $r = 0$.

2. If $|J_{it}^{s+}| \leq 1$, then go to step 9.

3. If $I^+_{j(i,t,s)t} = \emptyset$ and $I^+_{j'(i,t,s)t} = \emptyset$, then go to step 9.

4. For each (j, τ), with $\tau \leq t$ and $v_{it}^s > \mathcal{C}_{ijt}^s$, set $\pi_{j\tau} = \pi_{j\tau} + v_{it}^s - \mathcal{C}_{it}^{s-}$; set $v_{it}^s = \mathcal{C}_{it}^{s-}$.

5. (a) Set $I^+ = I^+_{j(i,t,s)t} \cup I^+_{j'(i,t,s)t}$ and execute the dual ascent procedure.
   (b) Set $I^+ = I^+ \cup \{(i,t,s)\}$ and execute the dual ascent procedure.
   (c) Set $I^+ = I \times \mathcal{T} \times \mathcal{S}$ and execute the dual ascent procedure.

6. If v_{it}^s is changed, then return to step 2.

7. Execute the primal procedure.

8. If neither $v_D^+ > v_D$ nor $v_P^+ < v_P$, then $r \leftarrow r+1$; otherwise $r \leftarrow 0$ and update v_D and v_P.

9. If $v_D \geq v_P$, or $r = r_{max}$ or $q = |I \times \mathcal{T} \times \mathcal{S}|$, then stop; otherwise $q \leftarrow q+1, (i,t,s) \leftarrow (i,t,s)_q$, and return to step 2.

3.2.4 Dual Adjustment Procedure for u_j

This procedure will try to increase and/or decrease the values of these dual variables. If $u_j > 0$ and $\pi_{jt} > 0, \forall\, t$, then u_j can be decreased by $\min_t\{\pi_{jt}, u_j\}$ units, the dual solution remains feasible, and consequently the dual objective function value increases. On the other hand, if there is a slack $\pi_{j\tau} = 0$, then it may be worth increasing the value of u_j which leads to an increase in the value of slacks $\pi_{jt}, \forall\, t$. In such cases, u_j and slacks π_{jt} are increased by $\mathcal{M}$ units, with $\max\{\mathcal{C}_{ijt}^s\} < \mathcal{M} < \infty$. Such an increase in the slacks' value will possibly allow the increase of dual variables v_{it}^s in such a way that this increase will compensate the decrease in the dual objective value (variables u_j have a negative coefficient in the dual objective function). Even if the dual objective function value does not improve, it will certainly never worsen, and the change in the dual solution can lead to a change in the corresponding primal solution, improving the primal objective function value.

In the following description of the procedure, I^+ is the input set to the dual ascent procedure that corresponds to variables v_{it}^s whose value can be increased. In addition, consider the sets J_{it}^{s*}, $\forall (i,t,s)$, defined in sub-subsection 3.2.3.

1. $j \leftarrow 1$.

2. $\Delta_j = \min\limits_{\tau}\{\pi_{j\tau}\}$. If $\Delta_j > 0$, then go to step 6, else set $\Delta_j = \mathcal{M}$.

3. $\pi_{j\tau} = \pi_{j\tau} + \Delta_j, \forall\, \tau; u_j = u_j + \Delta_j$.

4. Set $I^+ = \{(i,t,s) \in I \times \mathcal{T} \times \mathcal{S} : J_{it}^{s*} = \{j\}\}$ and execute the dual ascent procedure.
   Set $I^+ = I \times \mathcal{T} \times \mathcal{S}$ and repeat the dual ascent procedure.

5. $\Delta_j = \min\limits_{\tau}\{\pi_{j\tau}\}$.

6. $\Delta_j = \min\{\Delta_j, u_j\}$. If $\Delta_j > 0$, then $\pi_{j\tau} = \pi_{j\tau} - \Delta_j, \forall\, \tau; u_j = u_j - \Delta_j$.

7. If $j < |J|$, then $j \leftarrow j+1$ and return to step 2, otherwise stop.

4 Illustrative Examples

We shall illustrate the heuristic by two small examples. Real-world problems are typically much larger and provide more challenging situations. For the sake of simplicity, we consider problems with only two scenarios, both with $p^1 = 0.70$ and $p^2 = 0.30$, three time periods ($T = 3$), three potencial facility locations ($M = 3$) and four potencial customers ($N = 4$). In terms of the primal formulations, we are dealing with problems with *only* 81 decision variables and 99 restrictions.

Example 1

Consider the problem's data in tables 1–3: possible customers, assignment and fixed costs, respectively. We note that at $t = 1$ (present time) the input data is the same for both scenarios. In table 1 we can see that, under scenario 2, customer 1's demand's should not be considered in period $t = 3$ nor customer 4's demand's for periods $t > 1$.

The weighted assignment costs are presented in table 4. The initial dual solution and the initial slacks (derived after the weighting of the fixed costs) are shown in tables 5 and 6, respectively.

t		1	2	3
	1	(1,1)	(1,1)	(1,0)
i	2	(1,1)	(1,1)	(1,1)
	3	(1,1)	(1,1)	(1,1)
	4	(1,1)	(1,0)	(1,0)

Table 1: Possible customers, $(\delta^1_{it}, \delta^2_{it})$.

t		1			2			3		
j		1	2	3	1	2	3	1	2	3
	1	(5,5)	(7,7)	(10,10)	(7,10)	(8,9)	(13,14)	(9,–)	(8,–)	(19,–)
i	2	(10,10)	(6,6)	(6,6)	(11,12)	(7,7)	(8,11)	(12,11)	(7,7)	(10,13)
	3	(6,6)	(10,10)	(12,12)	(7,9)	(11,13)	(13,13)	(7,10)	(13,15)	(13,14)
	4	(4,4)	(7,7)	(12,12)	(6,–)	(10,–)	(14,–)	(7,–)	(11,–)	(14,–)

Table 2: Assignment costs, (c^1_{ijt}, c^2_{ijt}).

t	1			2			3		
$s \diagdown j$	1	2	3	1	2	3	1	2	3
1	7	8	$+\infty$	9	10	11	$+\infty$	11	12
2	7	8	$+\infty$	12	10	12	$+\infty$	15	12

Table 3: Fixed costs, f^s_{jt}.

	t		1			2			3		
	j		1	2	3	1	2	3	1	2	3
		1	3.5	4.9	7.0	4.9	5.6	9.1	6.3	5.6	13.3
$s = 1$	i	2	7.0	4.2	4.2	7.7	4.9	5.6	8.4	4.9	7.0
		3	4.2	7.0	8.4	4.9	7.7	9.1	4.9	9.1	9.1
		4	2.8	4.9	8.4	4.2	7.0	9.8	4.9	7.7	9.8
		1	1.5	2.1	3.0	3.0	2.7	4.2	–	–	–
$s = 2$	i	2	3.0	1.8	1.8	3.6	2.1	3.3	3.3	2.1	3.9
		3	1.8	3.0	3.6	2.7	3.9	3.9	3.0	4.5	4.2
		4	1.2	2.1	3.6	–	–	–	–	–	–

Table 4: Weighted assignment costs, $\mathcal{C}^s_{ijt}$.

t		1	2	3
	1	(3.5, 1.5)	(4.9, 2.7)	(5.6,–)
i	2	(4.2, 1.8)	(4.9, 2.1)	(4.9, 2.1)
	3	(4.2, 1.8)	(4.9, 2.7)	(4.9, 3.0)
	4	(2.8, 1.2)	(4.2,–)	(4.9,–)

Table 5: Initial dual solution, (v^1_{it}, v^2_{it}).

t		1	2	3
	1	7.0	9.9	$+\infty$
j	2	8.0	10.0	12.2
	3	$+\infty$	11.3	12.0

Table 6: Initial slacks, π_{jt}.

The dual ascent procedure tries to increase the variables v^s_{it} belonging to I^+, following an arbitrary sequence of these variables. We chose to consider the variables ordered by increasing values of t, s and i, respectively. We show below some of the first steps of the algorithm.

$(t, s) = (1, 1)$

$i = 1$:
$\min_j\{\pi_{j1} : v^1_{11} - \mathcal{C}^1_{1j1} \geq 0\} = \pi_{11} = 7\,,\ \Delta^1_{11} = \min\{7, 4.9 - 3.5\} = 1.4$, $\pi_{11} = 7 - 1.4 = 5.6$, $v^1_{11} = 3.5 + 1.4 = 4.9$;

$i = 2$:
$\min_j\{\pi_{j1} : v^1_{21} - \mathcal{C}^1_{2j1} \geq 0\} = \min_j\{\pi_{21}, \pi_{31}\} = 8\,,\ \Delta^1_{21} = \min\{8, 4.2 - 4.2\} = 0$, $v^1_{21} = 4.2$;

$i = 3$:
$\min_j\{\pi_{j1} : v^1_{31} - \mathcal{C}^1_{3j1} \geq 0\} = \pi_{11} = 5.6\,,\ \Delta^1_{31} = \min\{5.6, 7 - 4.2\} = 2.8$, $\pi_{11} = 5.6 - 2.8 = 2.8$, $v^1_{31} = 4.2 + 2.8 = 7$;

$i = 4$:
$\min_j\{\pi_{j1} : v^1_{41} - \mathcal{C}^1_{4j1} \geq 0\} = \pi_{11} = 2.8\,,\ \Delta^1_{41} = \min\{2.8, 4.9-2.8\} = 2.1$, $\pi_{11} = 2.8-2.1 = 0.7$, $v^1_{41} = 2.8 + 2.1 = 4.9$.

The algorithm proceeds to $(t,s) = (1,2)$, increasing v^2_{11} to 2.1 and v^2_{31} to 1.9. Afterwards, for $t = 2$ and $s = 1$, v^1_{12} is blocked by $\pi_{11} = 0$; for $i = 2$:

$\min_j\{\pi_{j\tau} : v^1_{22} - \mathcal{C}^1_{2j2} \geq 0,\ \tau \leq 2\} = \min\{\pi_{21}, \pi_{22}\} = \pi_{21} = 8\,,\ \Delta^1_{22} = \min\{8, 5.6-4.9\} = 0.7$, $\pi_{21} = 8 - 0.7 = 7.3$, $\pi_{22} = 10 - 0.7 = 9.3$, $v^1_{22} = 4.9 + 0.7 = 5.6$.

The dual ascent procedure continues until all the dual variables are blocked by some slack. At the end, we obtain the dual solution $\{v^{s+}_{it}\}$ and associated slacks $\{\pi^+_{jt}\}$ shown in tables 7 and 8, respectively. In addition, at the end of this procedure $u_j = 0$, $\forall j$. The corresponding dual objective function value is equal to $v^+_D = 87.8$.

With sets $J^* = \{(1,1), (2,1)\}$, $J^*_t = \{1,2\}$, $\forall t$, the primal procedure advances with sets $J^+ = J^*$ and $J^+_t = J^*_t, \forall\, t$. In fact, facilities 1 and 2 are both essencial for some customers at $t = 1$. For instance, $v^{1+}_{21} > \mathcal{C}^1_{221}$ but $v^{1+}_{21} < \mathcal{C}^1_{211}$, and $v^{2+}_{31} > \mathcal{C}^2_{311}$ but $v^{2+}_{31} < \mathcal{C}^2_{321}$, thus $t_1(j) = t_2(j) = 1$, $j = 1,2$. Then, $v^+_P = 87.8 = v^+_D$, which means that the optimal solution has been found (illustrated in figure 1). Despite the simplicity of this example, some of the inherent features of a nondeterministic and dynamic problem can be observed.

t		1	2	3
	1	(4.9, 2.1)	(4.9, 3.0)	(6.3,–)
i	2	(6, 1.8)	(5.6, 3.3)	(7.0, 3.3)
	3	(7, 1.9)	(4.9, 2.7)	(4.9, 3.0)
	4	(4.9, 1.2)	(4.2,–)	(4.9,–)

Table 7: Dual solution from the ascent procedure, $(v^{1+}_{it}, v^{2+}_{it})$.

t		1	2	3
	1	0	9.9	$+\infty$
j	2	0	3.8	8.2
	3	$+\infty$	11.3	12

Table 8: Slacks, π^+_{jt}.

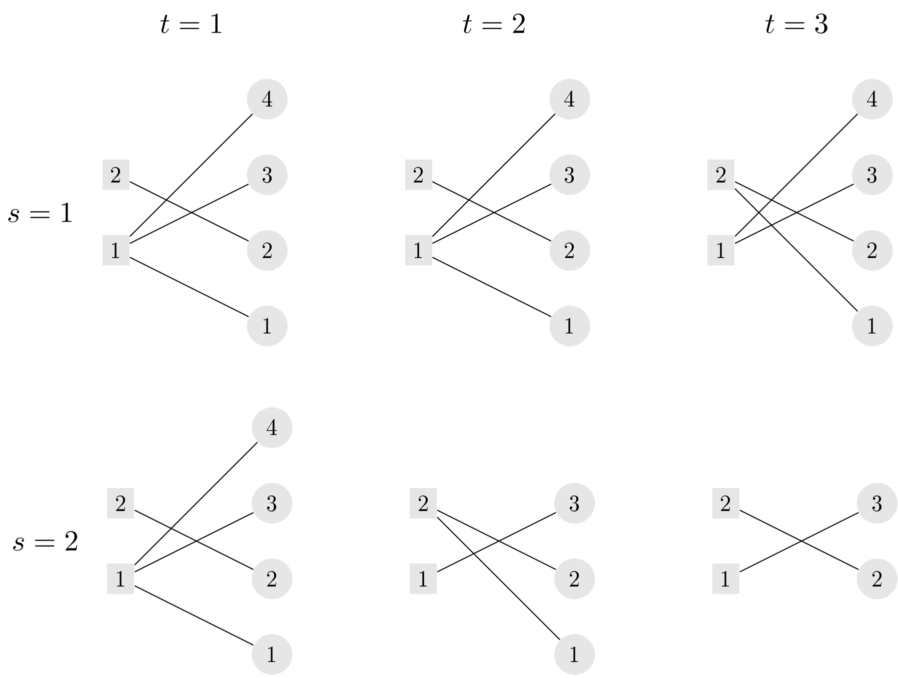

Figure 1: Optimal solution for example 1.

Example 2

Consider the problem's data in tables 9–11. As in the previous example, at $t = 1$ the input data is the same for both scenarios. The weighted assignment costs are presented in table 12. The initial dual solution and the initial slacks are shown in tables 13 and 14, respectively.

t		1	2	3
	1	(1,1)	(1,0)	(1,0)
i	2	(1,1)	(1,1)	(1,1)
	3	(1,1)	(1,1)	(1,1)
	4	(1,1)	(1,0)	(1,0)

Table 9: Possible customers, $(\delta^1_{it}, \delta^2_{it})$.

t		1			2			3		
j		1	2	3	1	2	3	1	2	3
	1	(5,5)	(8,8)	(10,10)	(7,–)	(9,–)	(11,–)	(9,–)	(12,–)	(12,–)
i	2	(8,8)	(5,5)	(6,6)	(11,8)	(6,7)	(7,9)	(13,13)	(7,8)	(10,12)
	3	(6,6)	(5,5)	(7,7)	(7,7)	(6,8)	(8,12)	(7,8)	(9,8)	(8,13)
	4	(4,4)	(6,6)	(8,8)	(6,–)	(7,–)	(9,–)	(7,–)	(8,–)	(9,–)

Table 10: Assignment costs, (c^1_{ijt}, c^2_{ijt}).

t	1			2			3		
$s \setminus j$	1	2	3	1	2	3	1	2	3
1	15	17	13	17	19	14	$+\infty$	20	15
2	15	17	13	18	19	15	$+\infty$	21	15

Table 11: Fixed costs, f^s_{jt}.

		t	1			2			3		
		j	1	2	3	1	2	3	1	2	3
		1	3.5	5.6	7.0	4.9	6.3	7.7	6.3	8.4	8.4
$s=1$	i	2	5.6	3.5	4.2	7.7	4.2	4.9	9.1	4.9	7.0
		3	4.2	3.5	4.9	4.9	4.2	5.6	4.9	6.3	5.6
		4	2.8	4.2	5.6	4.2	4.9	6.3	4.9	5.6	6.3
		1	1.5	2.4	3.0	–	–	–	–	–	–
$s=2$	i	2	2.4	1.5	1.8	2.4	2.1	2.7	3.9	2.4	3.6
		3	1.8	1.5	2.1	2.1	2.4	3.6	2.4	2.4	3.9
		4	1.2	1.8	2.4	–	–	–	–	–	–

Table 12: Weighted assignment costs, $\mathcal{C}^s_{ijt}$.

t		1	2	3
	1	(3.5, 1.5)	(4.9,–)	(6.3,–)
i	2	(3.5, 1.5)	(4.2, 2.1)	(4.9, 2.4)
	3	(3.5, 1.5)	(4.2, 2.1)	(4.9, 2.4)
	4	(2.8, 1.2)	(4.2,–)	(4.9,–)

Table 13: Initial dual solution, (v^1_{it}, v^2_{it}).

t		1	2	3
	1	15.0	17.3	$+\infty$
j	2	17.0	19.0	20.3
	3	13.0	14.3	15.0

Table 14: Initial slacks, π_{jt}.

After the dual ascent procedure, we obtain the dual solution and associated slacks shown in tables 15 and 16, respectively. At the end of this procedure $u_j = 0, \forall j$. We can see that all dual variables belonging to I^+ were increased, except the one corresponding to the pseudo customer $(i, t, s) = (3, 3, 2)$. The corresponding dual objective function value is equal to $v^+_D = 94.4$.

With sets $J^* = \{(1,1), (2,1)\}$, $J^*_t = \{1,2\}, \forall t$, the primal procedure advances with sets $J^+ = J^*$ and $J^+_t = J^*_t, \forall t$. Facilities 1 and 2 are both essential at $t = 3$, then $t_1(1) = t_1(2) = 3$ and $t_2(1) = t_2(2) = 1$. The primal objective function value equals $v^+_P = 98.5 > v^+_D$, so the heuristic continues to the primal-dual adjustment procedure.

t		1	2	3
	1	(7, 3)	(6.3,–)	(8.4,–)
i	2	(5.6, 2.4)	(7.7, 2.4)	(8.1, 3.6)
	3	(4.9, 1.8)	(4.9, 2.4)	(5.6, 2.4)
	4	(5.6, 1.8)	(4.9,–)	(5.6,–)

Table 15: Dual solution from the ascent procedure, $(v_{it}^{1+}, v_{it}^{2+})$.

t		1	2	3
	1	0.0	11.4	$+\infty$
j	2	0.0	10.1	15.9
	3	7.1	10.4	13.9

Table 16: Slacks, π_{jt}^{+}.

The previous result means that at least one of the conditions (3.19) is violated. For instance, $v_{11}^{1+} > \mathcal{C}_{1j1}^{1}$, for $j = 1, 2$, thus $|J_{11}^{1+}| = 2$.

The best source and the second-best source for pseudo costumer $(i, t, s) = (1, 1, 1)$ are, respectively, $j(1, 1, 1) = 1$ and $j'(1, 1, 1) = 2$. In addition, $I_{11}^{+} = \{(3, 3, 1)\}$ and $I_{21}^{+} = \{(2, 3, 1), (2, 3, 2)\}$. Within the primal-dual adjustment procedure, slacks π_{11}^{+} and π_{21}^{+} are increased $v_{11}^{1+} - \mathcal{C}_{11}^{1-} = 7 - 5.6 = 1.4$ units and v_{11}^{1+} is decreased to $\mathcal{C}_{11}^{1-} = 5.6$. After the dual ascent procedures, initially with $I^{+} = \{(3, 3, 1), (2, 3, 1), (2, 3, 2)\}$, no further improvements are possible. The resulting dual solution is presented in table 17, with associated slacks presented in table 18. The dual objective function value is updated to $v_D = 95.1$.

t		1	2	3
	1	(5.6, 3)	(6.3,–)	(8.4,–)
i	2	(5.6, 2.4)	(7.7, 2.4)	(**9.2**, **3.9**)
	3	(4.9, 1.8)	(4.9, 2.4)	(**6.3**, 2.4)
	4	(5.6, 1.8)	(4.9,–)	(5.6,–)

Table 17: Dual solution after the dual ascent procedures within the primal-dual adjustment procedure.

t		1	2	3
	1	0.6	10.6	$+\infty$
j	2	0.0	8.7	14.5
	3	5	8.3	11.8

Table 18: Slacks after the dual ascent procedures within the primal-dual adjustment procedure.

From the primal procedure results $J^{*} = J^{+} = \{(2, 1)\}$, and $J_t^{+} = \{2\}$, $\forall t$, then $v_P = 95.1 = v_D$, which means that the heuristic found the optimal solution.

5 Conclusions and Future Work

We propose a primal-dual heuristic approach for a simple dynamic facility location problem with uncertainty. From the tests done so far, the heuristic is capable of calculating the optimal solution most of the time. Nevertheless, more computational tests need to be done not only to assess the heuristic's ability to calculate the optimal solution, but also to compare its behavior with the behavior of a general solver (namely in what concerns computational times). In situations where it is not possible to find the optimal solution, it is possible to consider both local search procedures and also the use of this heuristic within a branch and bound procedure (similar to Dias et al. (2007)). We also intend to incorporate into our model the possibility of closing existing facilities during the planning horizon. As time goes by, the uncertainty associated with future time periods that become the present disappears, so this dynamic model should be applied considering a rolling time window.

References

Cornuejols, G., Nemhauser, G. L., and Wolsey, L. A. (1990). The uncapacitated facility location problem. In Mirchandani, P. B. and Francis, R. L., editors, *Discrete Location Theory*, pages 119–171. John Wiley&Sons.

Current, J., Ratick, S., and ReVelle, C. (1997). Dynamic facility location when the total number of facilities is uncertain: A decision analysis approach. *European Journal of Operational Research*, 110:597–609.

Dias, J., Captivo, M. E., and Clímaco, J. (2007). Efficient primal-dual heuristic for a dynamic location problem. *Computers & Operations Research*, 34(6):1800–1823.

Erlenkotter, D. (1978). A dual-based procedure for uncapacitated facility location. *Operations Research*, 26(6):992–1009.

Erlenkotter, D. (1981). A comparative study of approaches to dynamic location problems. *European Journal of Operational Research*, 6(2):133–143.

Jornsten, K. and Bjorndal, M. (1994). Dynamic location under uncertainty. *Studies in Regional and Urban Planning*, 3:163–183.

Laporte, G., Louveaux, F. V., and Hamme, L. V. (1994). Exact solution to a location problem with stochastic demands. *Transportation Science*, 28(2):95–103.

Lin, C. K. Y. (2009). Stochastic single-source capacitated facility location model with service level requirements. *International Journal of Production Economics*, 117(2):439–451.

Louveaux, F. V. (1993). Stochastic location analysis. *Location Science*, 1:127–154.

Louveaux, F. V. and Peeters, D. (1992). A dual-based procedure for stochastic facility location. *Operations Research*, 40(3):564–573.

Ravi, R. and Sinha, A. (2004). Hedging uncertainty: approximation algorithms for stochastic optimization problems. *Lecture Notes in Computer Science*, 3064:101–115.

Rockafellar, R. T. and Wets, R. J.-B. (1991). Scenarios and policy aggregation in optimization under uncertainty. *Mathematics of Operations Research*, 16:119–147.

Romauch, M. and Hartl, R. F. (2005). Dynamic facility location with stochastic demands. *Lecture Notes in Computer Science*, 3777:180–189.

Serra, D. and Marianov, V. (1998). The p-median problem in a changing network: the case of Barcelona. *Location Science*, 6:383–394.

Snyder, L. V. (2006). Facility location under uncertainty: a review. *IIE Transactions*, 38:537–554.

Van Roy, T. J. and Erlenkotter, D. (1982). A dual-based procedure for dynamic facility location. *Management Science*, 28(10):1091–1105.